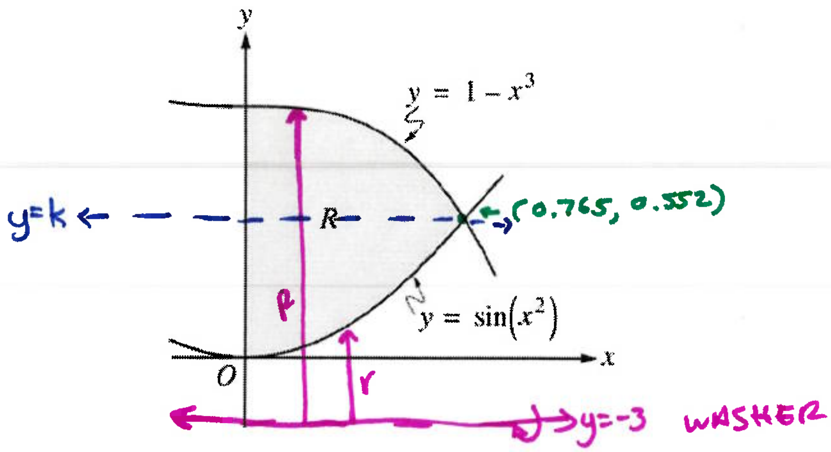

1. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = \underbrace{1 - x^3}_{f_1}$ and $\underbrace{y = \sin(x^2)}_{f_2}$, as shown in the figure above.

(a) Find the area of R.

$$\text{Area of } R = \int_0^{0.765} \underbrace{\left(1 - x^3 - \sin(x^2)\right)}_{\text{1pt: integrand}} dx$$

$$= 0.534$$

1pt: answer

1pt: Limits correct in (a), (b), or (c)

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Continue problem 1 on page 5.

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin(x^2)$ intersect. Find k and determine whether this line divides R into two regions of equal area. Show the work that leads to your conclusion.

y-value of intersection pt.

$k = 0.552$

Area above $y=k$ $= \int_0^{0.765} (1 - x^3 - k)\,dx$

$= 0.257$

1pt: integral(s) w/ k-value

1pt: value(s) of integral(s)

Area below $y=k$ $= \int_0^{0.765} (k - \sin(x^2))\,dx$

$= 0.277$

1pt: conclusion tied to part (a) or comparison of the two integrals

The two regions are not equal areas.

(c) Find the volume of the solid generated when R is revolved about the line $y = -3$.

$\text{Volume} = \pi \int \left((\text{outside radius})^2 - (\text{inside radius})^2 \right) dx$ ... ☺

$$\text{Volume} = \pi \int_0^{0.765} \left[\underbrace{(1 - x^3 - -3)^2 - (\sin(x^2) - -3)^2}_{\text{2pts: integrand}} \right] dx$$

$= 11.841$ ← 1pt: answer

GO ON TO THE NEXT PAGE.

2. The penguin population on an island is modeled by a differentiable function P of time t, where $P(t)$ is the number of penguins and t is measured in years, for $0 \le t \le 40$. There are 100,000 penguins on the island at time $t = 0$. The birth rate for the penguins on the island is modeled by

then rate $P'(t)$

derivative $\rightarrow$ $$B(t) = 1000e^{0.06t} \text{ penguins per year}$$

and the death rate for the penguins on the island is modeled by

derivative $\rightarrow$ $$D(t) = 250e^{0.1t} \text{ penguins per year.}$$

(a) What is the rate of change of the penguin population on the island at time $t = 0$?

$P'(t) =$ rate born $-$ rate die

$P'(0) = B(0) - D(0)$

$= 750$ penguins/yr

1pt : answer

(b) To the nearest whole number, what is the penguin population on the island at time $t = 40$?

$P(40) =$ initial penguins $+ \int(\text{rate born} - \text{rate die}) \cdot$

$$P(40) = 100{,}000 + \int_0^{40} \underbrace{(B(t) - D(t))}\,dt$$

$= 133{,}057.5605$

$\approx 133{,}058$ penguins

1 pt : limits

1 pt : integrand

1pt : answer

Continue problem 2 on page 7.

(c) To the nearest whole number, what is the average rate of change of the penguin population on the island for $0 \le t \le 40$?

$\underbrace{\text{average rate of change of the penguin population}}_{P'(t) = \frac{P(b)-P(a)}{b-a}}$

$$\text{avg rate of change of pop.} = \frac{P(40) - P(0)}{40 - 0}$$

$$= \frac{133058 - 100000}{40}$$

$$= 826.450$$

$$\approx 826 \text{ penguins/yr}$$

1pt : answer

crit #s + endpts

(d) To the nearest whole number, find the absolute minimum penguin population and the absolute maximum penguin population on the island for $0 \le t \le 40$. Show the analysis that leads to your answers.

$P'(t) = B(t) - D(t) = 0$ ← 1pt : $B(t) - D(t) = 0$

$B(t) = D(t)$

$t = 34.657$ ← 1pt : solves for t

$P(0) = 100{,}000$

$P(40) \approx 133058$

$$P(34.657) = 100000 + \int_0^{34.657} (B(t) - D(t))\,dt$$

$$= 139166.667$$

1pt : max value

1pt : min value

The min population is 100,000 and the max population is 139,167 penguins.

NO CALCULATOR ALLOWED

$\Delta t = 10$ $\Delta t = 12$ $\Delta t = 8$

t (days)	0	10	22	30
$W'(t)$ (GL per day)	L 0.6	L 0.7	L 1.0	0.5

3. The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in gigaliters (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \le t \le 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

(a) Use the tangent line approximation to W at time $t = 30$ to predict the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

$y - y_1 = m(x - x_1)$

$W - W_1 = W'(t)(t - t_1)$

$W(32) - W(30) = W'(30)(32 - 30)$

$W(32) - 125 = 0.5(32 - 30)$

$W(32) = 0.5(32 - 30) + 125$

$= 126$ gigaliters

1pt: answer

(b) Use a left Riemann sum, with the three subintervals indicated by the data in the table, to approximate $\int_0^{30} W'(t)\,dt$. Use this approximation to estimate the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 0$. Show the computations that lead to your answer.

need $W(0)$

$\int_0^{30} W'(t)\,dt \approx 10(0.6) + 12(0.7) + 8(1.0)$

1pt: left sum

$= 22.4$ gigaliters

1pt: approx

$\int_0^{30} W'(t)\,dt = W(t)\Big|_0^{30}$

$\int_0^{30} W'(t)\,dt = W(30) - W(0)$ ← isolate $W(0)$ ☺

$\therefore,\ W(0) = W(30) - \int_0^{30} W'(t)\,dt$

$= 125 - 22.4$

1pt: answer

$= 102.6$ gigaliters

Continue problem 3 on page 15.

(c) Explain why there must be at least one time t, other than $t = 10$, such that $W'(t) = 0.7$ GL/day.

W' is diff'able → ∴, W' is continuous

$$\left.\begin{array}{l} W'(22) = 1.0 \\ W'(30) = 0.5 \end{array}\right\} \text{0.7 is btn 1.0 \& 0.5}$$

By IVT, there must be at least one time t, other than $t=10$, on $[22, 30]$ s.t. $W'(t) = 0.7$ b/c $W'(30) < 0.7 < W'(22)$. and W' is cont.

2pts : explanation

(d) The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometers per day, with respect to t when $t = 30$ days.

derivative

$\left.\frac{dA}{dt}\right|_{t=30} = ?$

$A = 0.3W^{2/3}$

$\frac{dA}{dt} = 0.3\left(\frac{2}{3}W^{-1/3}\right)\frac{dW}{dt}$

$\left.\frac{dA}{dt}\right|_{t=30} = 0.3\left(\frac{2}{3}\cdot(125)^{-1/3}\right)\cdot(0.5)$

$= 0.02$

W(30) initial condition

$\left.\frac{dW}{dt}\right|_{t=30} = W'(30) = 0.5$ see table

2pts : $\frac{dA}{dt}$

1pt : answer

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4. Let f be the function given by $f(x) = (x^2 - 2x - 1)e^x$.

(a) Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$\lim_{x \to \infty} f(x) =$ ∞

1pt: answers

$\lim_{x \to -\infty} f(x) =$ 0

(b) Find the intervals on which f is increasing. Show the analysis that leads to your answer.

↳ $f' > 0$

$f'(x) = e^x(2x-2) + (x^2-2x-1)e^x$ ← 2pts: $f'(x)$

$= e^x(2x-2+x^2-2x-1)$

$= e^x(x^2-3)$

$0 = e^x(x^2-3)$

$e^x = 0$ no x-value works

$x^2 - 3 = 0$

$x = \pm\sqrt{3}$

f': $+$ | $-$ | $+$

-5 $\quad -\sqrt{3}$ $\quad 0$ $\quad \sqrt{3}$ $\quad 5$

1pt: intervals

1pt: reason

f is inc on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ b/c $f' > 0$ on those intervals

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Find the intervals on which the graph of f is concave down. Show the analysis that leads to your answer.

$\hookrightarrow f'' < 0$

$f''(x) = (x^2-3)e^x + e^x(2x)$ ← 2pts: $f''(x)$

$= e^x(x^2+2x-3)$

$= e^x(x+3)(x-1)$

$\cancel{0 = e^x}$ $x+3=0$ $x-1=0$

$x=-3$ $x=1$

f'': + − + ; -4, -3, 0, 1, 2

1pt: interval

1pt: reason

f is concave down on $(-3, 1)$ b/c $f'' < 0$ on $(-3, 1)$

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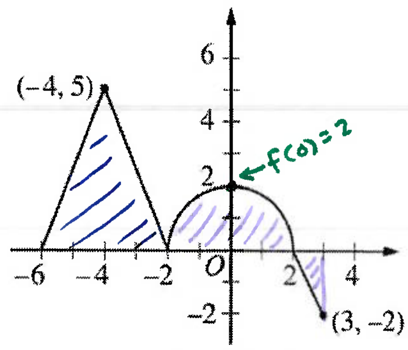
NO CALCULATOR ALLOWED



Graph of f

5. The graph of the continuous function f, consisting of three line segments and a semicircle, is shown above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$.

(a) Find $g(-6)$ and $g(3)$.

$g(-6) = \int_{-2}^{-6} f(t)\,dt$

$= -\int_{-6}^{-2} f(t)\,dt$

$= -\left(\frac{1}{2}(4)(5)\right)$

$\boxed{g(-6) = -10}$

Area Δ

$g(3) = \int_{-2}^{3} f(t)\,dt$

Area semicircle and Δ

ok to stop here for AP → $= \frac{1}{2}\pi(2)^2 + \frac{1}{2}(1)(-2)$

$\boxed{g(3) = 2\pi - 1}$

1pt: $g(-6)$

1pt: $g(3)$

(b) Find $g'(0)$.

$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$

$g'(x) = f(x)$

$g'(0) = f(0)$

$\boxed{g'(0) = 2}$

1pt: $g'(0)$

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a horizontal tangent. Determine whether g has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

$\to g'(x)=0$

↳ g' pos to neg ↳ g' neg to pos

$g'(x) = f(x) = 0$
@ $x = -6, -2, 2$
↑ not in interval $(-6,3)$ so only $x=-2$, and $x=2$

1 pt: horizontal tangents @ $x=-2$ and $x=2$

g does not have rel. max or min @ $x=-2$ b/c
g' does not change signs @ $x=-2$

2 pts: answers w/ reasons

g has rel. max @ $x=2$ b/c
g' changes from pos to neg @ $x=2$.

(d) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a point of inflection. Explain your reasoning.

$g''=0$ or DNE p.i.p's

g'' changes signs

$g'(x) = f(x)$
$g''(x) = f'(x) = 0$ or DNE
@ $x=-4$, $x=-2$, $x=0$, $x=2$ ← possible inf. pts.

2 pts: values of x

g has inf pts @ $x=-4$, $x=-2$, and $x=0$
b/c g'' changes signs at these x-values.

1 pt: reason

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NO CALCULATOR ALLOWED

6. Let f be a function with $f(2) = -8$ such that for all points (x, y) on the graph of f, the slope is given by $\frac{3x^2}{y}$.

(a) Write an equation of the line tangent to the graph of f at the point where $x = 2$ and use it to approximate $f(1.8)$.

$y - y_1 = m(x - x_1)$

$m = \frac{3x^2}{y}$

$\left.\frac{dy}{dx}\right|_{(2,-8)} = \frac{3(2)^2}{-8} = -\frac{3}{2}$

1pt - slope

$y - -8 = -\frac{3}{2}(x-2)$

1pt tangent line → or $y + 8 = -\frac{3}{2}(x-2)$

$y + 8 = -\frac{3}{2}(1.8-2)$

$y = -\frac{3}{2}(1.8-2) - 8$

$f(1.8) \approx -\frac{3}{2}(1.8-2) - 8$ ← ok to stop here for AP

≈ -7.7

↖ 1pt approximation

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Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

(b) Find an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = \frac{3x^2}{y}$ with the initial condition $f(2) = -8$.

↳ solve diff. eq by separate variable x w/dx, y w/dy

$dy = \frac{3x^2}{y}dx$

$\int y\,dy = \int 3x^2\,dx$ → 1pt: separate variables

2pts: antideriv →

$\frac{1}{2}y^2 = x^3 + C$ ← initial condition (2, −8)

1pt: "+C"

$\frac{1}{2}(-8)^2 = 2^3 + C$

$32 = 8 + C$

$24 = C$

1pt use initial condition

$\frac{1}{2}y^2 = x^3 + 24$

$y^2 = 2x^3 + 48$

$y = \pm\sqrt{2x^3+48}$

$y = -\sqrt{2x^3+48}$ ← 1pt: solve for y

keep neg. b/c $y<0$, $-8<0$ ☺

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