t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

$\frac{v(b)-v(a)}{b-a}$

$$v'(16) = \frac{v(20)-v(12)}{20-12} = \frac{240-200}{20-12} = 5 \text{ m/min}^2$$

1pt: approximation (units not required)

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

$$\int_0^{40} |v(t)|\, dt = 150(16) + 220(4) + 240(8) + 200(12)$$

ok to stop here

$$= 7600 \text{ meters}$$

simplify not required

1pt: Right sum
1pt: approx.

$\int_0^{40} |v(t)|\, dt$ is the total distance Johanna jogs, in meters, from $t=0$ to 40 minutes

1pt: explanation (with units)

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Continue problem 3 on page 15.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by

velocity → $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

→ $a(t) = v'(t)$

Bob $a(t) \rightarrow B'(t) = 3t^2 - 12t$

Bob's acceleration @ $t=5$ $= B'(5)$

1pt: uses $B'(t)$

$= 3(5)^2 - 12(5)$ ← ok to stop here

1pt: answer (units optional)

$= 15 \text{ m/min}^2$

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

$\frac{1}{b-a}\int_a^b v(t)$

Bob avg velocity $= \frac{1}{10-0}\int_0^{10} B(t)\,dt$

1pt: integral (w/ limits)

$= \frac{1}{10}\int_0^{10}(t^3 - 6t^2 + 300)\,dt$

1pt: antiderivative

$= \frac{1}{10}\left(\frac{1}{4}t^4 - 2t^3 + 300t\right)\Big|_0^{10}$

$= \frac{1}{10}\left(\frac{1}{4}(10)^4 - 2(10)^3 + 300(10) - 0\right)$ ← ok to stop here

1pt: answer (units optional)

$= \frac{1}{10}\left(\frac{10000}{4} - 2(1000) + 3000\right)$

$= \frac{1000}{4} - 200 + 300$

$= 250 - 200 + 300$

$= 350 \text{ m/min}$

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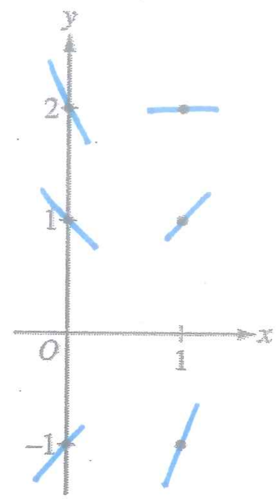
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

y \ x	0	1
2	-2	0
1	-1	1
-1	1	3



1pt: slopes where x=0

1pt: slopes where x=1

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in $\boxed{\text{Quadrant II}}$. Give a reason for your answer.

→ 2nd drivative

→ f" >0 or <0 in Quad II

$\frac{dy}{dx} = 2x - y$

$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$= 2 - (2x - y)$

$= 2 - 2x + y$

1pt: $\frac{d^2y}{dx^2}$

In Quad II, $x < 0$ and $y > 0$

$\frac{d^2y}{dx^2} > 0$ in quad II,

$\therefore$, all solution curves are concave up. in Quad II.

1pt: concave up w/ reason

Continue problem 4 on page 17.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

crit # changes pos to neg or neg to pos

crit #, if $\left.\frac{dy}{dx}\right|_{(2,3)} = 0$ or DNE

$$\left.\frac{dy}{dx}\right|_{(2,3)} = 2(2) - 3$$

$$= -1$$

1pt: consider $\left.\frac{dy}{dx}\right|_{(2,3)}$

f does not have rel. max nor min @ $x=2$

b/c $\left.\frac{dy}{dx}\right|_{(2,3)} \neq 0$ or DNE...

1pt: answer w/ reason

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y$

$y = mx + b$

$\frac{dy}{dx} = m$

$2x - y = m$

$2x - (mx + b) = m$

$2x - mx - b = m$

1pt: $\frac{dy}{dx} = m$

1pt: sets $2x - y = m$

$2 - m = 0 \qquad -b = m$

$2 = m$

$-b = 2$

$b = -2$

1pt: answer

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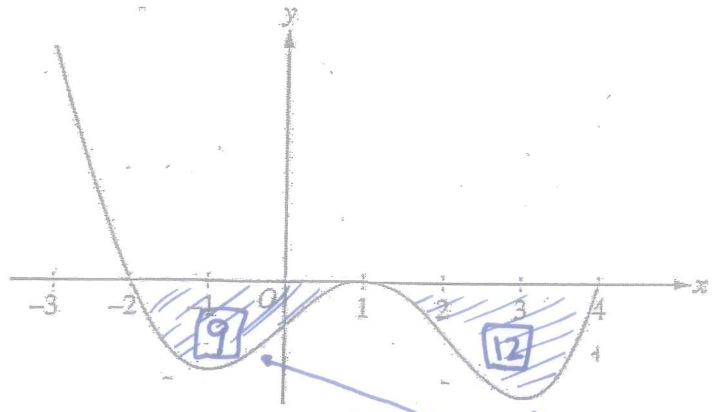

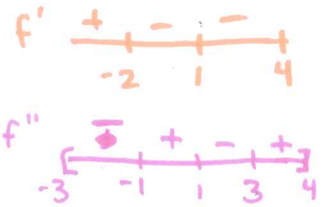

Graph of f'

5. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.

↳ f' changes pos to neg.

f has rel. max @ x = -2

b/c f' changes from pos to neg

@ x = -2

1pt: identifies x = -2

1pt: answer w/reason

(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

↳ f'' < 0 ↳ f' < 0

$f' < 0$ on $(-2, 1) \cup (1, 4)$

$f'' < 0$ on $(-3, -1) \cup (1, 3)$

f both concave down + decreasing on $(-2, -1) \cup (1, 3)$ b/c

$f'' < 0$ and $f' < 0$ on $(-2, -1) \cup (1, 3)$

1pt: intervals

1pt: reason

Continue problem 5 on page 19.

(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.

↳ f'' ~~changes signs~~

f has inf pts @ $x = -1$, and $x = 1$, and $x = 3$

b/c f'' changes signs @ $x = -1, 1$ and 3.

1pt: identifies $x = 1, -1, 3$

1pt: reason

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(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

$$f(x) = f(1) + \int_1^x f'(t)\,dt$$

1pt: integrand

1pt: expression for $f(x)$

$$f(4) = f(1) + \int_1^4 f'(t)\,dt$$

$= 3 + (-12)$ ← ok to stop here

$= -9$

1pt: $f(4)$ and $f(-2)$

$$f(-2) = f(1) + \int_1^{-2} f'(t)\,dt$$

$= 3 - \int_{-2}^{1} f'(t)\,dt$

$= 3 - (-9)$ ← ok to stop here

$= 12$

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6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

$y - y_1 = m(x - x_1)$

(x_1, y_1)

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$$

1pt: slope

1pt: equation of tangent line

$$y - 1 = \frac{1}{4}(x - -1)$$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

$\frac{dy}{dx}$ DNE

$\frac{dy}{dx} = \frac{1}{0}$

$$\frac{dy}{dx} = \frac{y}{3y^2 - x} = \frac{1}{0}$$

$3y^2 - x = 0$

$3y^2 = x$

1pt: $3y^2 - x = 0$

$y^3 - xy = 2$ ← given curve

$y^3 - 3y^2(y) = 2$

$y^3 - 3y^3 = 2$

$-2y^3 = 2$

$y^3 = -1$

$y = -1$

1pt: equation in one variable

$3(-1)^2 = x$

$3 = x$

1pt: coordinates

vertical line tangent to curve @ $(3, -1)$

Continue problem 6 on page 21.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

→ 2nd derivative

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

(quotient rule + implicit) ... ☺

$$\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\left(\frac{dy}{dx}\right) - (y)\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$$

2pts: implicit

1pts: subs in dy/dx value

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,1)} = \frac{(3(-1)^2 - (-1))\left(\frac{1}{4}\right) - (1)\left(6(1)\left(\frac{1}{4}\right) - 1\right)}{\left[3(1)^2 - (-1)\right]^2}$$

← ok to stop here

1pt: answer

$$= \frac{4\left(\frac{1}{4}\right) - \left(\frac{6}{4} - 1\right)}{4^2}$$

$$= \frac{1 - \frac{1}{2}}{16}$$

$$= \frac{\frac{1}{2}}{16}$$

$$= \frac{1}{32}$$

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