Topic 5: Differential Equations

Differential equations are tested every year. The actual solving of the differential equation is usually the main part of the problem, but it is accompanied by a question about its slope field or a tangent line approximation of some sort or something related. BC students may also be asked to approximate using Euler's Method. Large parts of the BC questions are often suitable for AB students and contribute to the AB subscore of the BC exam.

What students should be able to do:

- Find the *general solution* of a differential equation using the method of separation of variables (this is the *only* method tested).
- Find a *particular solution* using the initial condition to evaluate the constant of integration – initial value problem (IVP)
- Understand that proposed solution of a differential equation is a function (not a number) and if it and its derivative are substituted into the given differential equation the resulting equation is true. This may be part of doing the problem even if solving the differential equation is not required (see 2002 BC 5(c))
- Growth-decay problems.
- Draw a slope field by hand.
- Sketch a *particular solution* on a (given) slope field.
- Interpret a slope field.
- For BC only: Use Euler's Method to approximate a solution.
- For BC only: use the method of partial fractions to find the antiderivative after separating the variables.
- For BC only: understand the logistic growth model, its asymptotes, meaning, etc. The exams have never asked students to actually solve a logistic equation IVP.

NO CALCULATOR ALLOWED

$f'(3) = 3\sqrt{f(3)}$

Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x, where $f(3) = 25$.

(a) Find $f''(3)$.

$f'(x) = x(f(x))^{1/2}$

$$f''(x) = (f(x))^{1/2}(1) + x\left[\tfrac{1}{2}(f(x))^{-1/2} \cdot f'(x)\right]$$

$$= \sqrt{f(x)} + \tfrac{1}{2}x \cdot \frac{1}{\sqrt{f(x)}} \cdot f'(x)$$

2pts: $f''(x)$ (−2 if product or chain rule error)

$$f''(3) = \sqrt{f(3)} + \tfrac{1}{2} \cdot 3 \cdot \frac{1}{\sqrt{f(3)}} \cdot f'(3)$$

$$= \sqrt{25} + \tfrac{1}{2} \cdot 3 \cdot \frac{1}{\sqrt{25}} \cdot 3\sqrt{25}$$

ok to stop here

1pt: value @ $x=3$

$$= 5 + \tfrac{1}{2} 3 \cdot \tfrac{1}{5} \cdot 3 \cdot 5$$

$$= 5 + \frac{9}{2}$$

$$= \frac{10}{2} + \frac{9}{2}$$

$$= \frac{19}{2}$$

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

(b) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = x\sqrt{y}$ with the initial condition $f(3) = 25$.

y's w/ dy, x's w/ dx

$\frac{dy}{dx} = x\sqrt{y}$

$\int \frac{1}{\sqrt{y}}\,dy = \int x\,dx$ 1pt: separate variables

$\int y^{-1/2}\,dy = \int x\,dx$

$2y^{1/2} = \frac{1}{2}x^2 + C$ 1pt: antiderive y, 1pt: antiderive x, 1pt: "+ C"

$2(25)^{1/2} = \frac{1}{2}(3)^2 + C$ 1pt: initial condition

$10 = \frac{9}{2} + C$

$\frac{11}{2} = C$

$2y^{1/2} = \frac{1}{2}x^2 + \frac{11}{2}$

$y^{1/2} = \frac{1}{4}x^2 + \frac{11}{4}$

$y = \left(\frac{1}{4}x^2 + \frac{11}{4}\right)^2$ 1pt: solves for y.

END OF EXAMINATION

THE FOLLOWING INSTRUCTIONS APPLY TO THE BACK COVER OF THIS SECTION II BOOKLET.

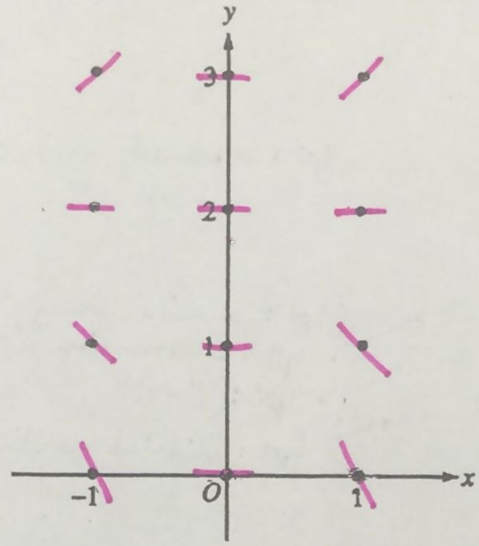
- MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE BACK OF THIS SECTION II BOOKLET.
- CHECK TO SEE THAT YOUR AP NUMBER APPEARS IN THE BOX(ES) ON THE BACK COVER.
- MAKE SURE THAT YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMINATIONS YOU HAVE TAKEN THIS YEAR.

NO CALCULATOR ALLOWED

5. Consider the differential equation $\frac{dy}{dx} = x^4(y-2)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.

y \ x	-1	0	1
3	1	0	1
2	0	0	0
1	-1	0	-1
0	-2	0	-2



1pt: zero slopes

1pt: positive + negative slopes

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

when $y < 2$ and $x \neq 0$

slope are neg @ pts. (x, y)
where $x \neq 0$ and $y < 2$

1pt: description

Continue problem 5 on page 13.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.

x's w/ dx, y's w/ dy

$$\frac{dy}{dx} = x^4(y-2)$$

$$\int \frac{1}{y-2}dy = \int x^4\,dx$$

1pt: separate variables

$u = y-2$

$du = dy$

$$\int \frac{1}{u}dy = \int x^4\,dx$$

$$\ln|u| = \frac{1}{5}x^5 + C$$

2pts: antiderivatives

1pt: "+C"

$$\ln|y-2| = \frac{1}{5}x^5 + C$$

$$\ln|0-2| = \frac{1}{5}(0)^5 + C$$

1pt: initial condition

$$\ln|-2| = C$$

$$\ln 2 = C$$

$$\ln|y-2| = \frac{1}{5}x^5 + \ln 2$$

$$|y-2| = e^{\frac{1}{5}x^5 + \ln 2}$$

$$y-2 = \pm e^{\frac{1}{5}x^5 + \ln 2}$$

neg. b/c $y-2<0$ when $y=0$ $0-2<0$ $-2<0$

$$y = -e^{\frac{1}{5}x^5 + \ln 2} + 2$$

1pt: solve for y

or $$y = -e^{1/5x^2} \cdot e^{\ln 2} + 2$$

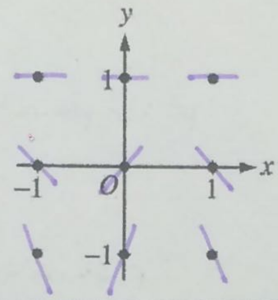
$$y = -2e^{\frac{1}{5}x^2} + 2$$

GO ON TO THE NEXT PAGE.

5. Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

y \ x	-1	0	1
1	0	0	0
0	-1	1	-1
-1	-4	4	-4



1pt: zero slopes

1pt: all other slopes

(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

slope $\rightarrow \frac{dy}{dx} = 0$

$\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$

$0 = (y-1)^2 \cos(\pi x)$

$(y-1)^2 = 0$ $\quad\quad$ $\cos(\pi x) = 0$

$y - 1 = 0$ $\quad\quad$ $\pi x = \pi/2$

$y = 1$ $\quad\quad$ $x = 1/2$

need $y = \#$ horizontal line.

$y = c$!

$\therefore, c = 1$

1pt: $c = 1$

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Continue problem 5 on page 13.

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(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

↳ y's w/ dy, x's w/ dx

$\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$

$\int \frac{1}{(y-1)^2}\, dy = \int \cos(\pi x)\, dx$

$\int (y-1)^{-2}\, dy = \int \cos(\pi x)\, dx$

1pt: separate variables

$w = \pi x$

$\frac{dw}{dx} = \pi$

$\frac{dw}{\pi} = dx$

$u = y-1$

$\frac{du}{dy} = 1$

$du = dy$

$\int u^{-2}\, du = \int \cos w \cdot \frac{dw}{\pi}$

$-u^{-1} = \frac{1}{\pi} \sin w + C$

$-(y-1)^{-1} = \frac{1}{\pi} \sin(\pi x) + C$

2pts: antiderive

1pt: "+C"

$-\frac{1}{0-1} = \frac{1}{\pi} \sin(\pi) + C$ ← use $f(1)=0$

1pt: use initial condition

$1 = C$

$-\frac{1}{y-1} = \frac{1}{\pi} \sin(\pi x) + 1$

$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + \frac{\pi}{\pi}$

$-\frac{1}{y-1} = \frac{\sin(\pi x) + \pi}{\pi}$

$\frac{1}{y-1} = -\frac{\sin(\pi x) + \pi}{\pi}$

$y-1 = -\frac{\pi}{\sin(\pi x) + \pi} \rightarrow y = -\frac{\pi}{\sin(\pi x) + \pi} + 1$

1pt: solves for y correctly

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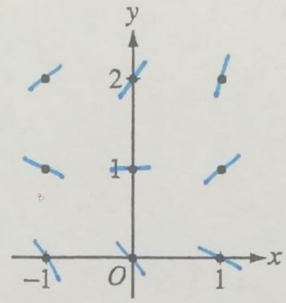
GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.

$y \backslash x$	-1	0	1
2	$1/2$	1	$3/2$
1	$-1/2$	0	$1/2$
0	$-3/2$	-1	$-1/2$



2pts: signs of slopes + steepness of slopes

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

$\frac{d^2y}{dx^2} > 0$

$$\frac{dy}{dx} = \frac{1}{2}x + y - 1$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx}$$

$$= \frac{1}{2} + \frac{1}{2}x + y - 1$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}x + y - \frac{1}{2}$$

2pts: $\frac{d^2y}{dx^2}$

$$\frac{1}{2}x + y - \frac{1}{2} > 0$$

$$y > -\frac{1}{2}x + \frac{1}{2}$$

solutions concave up when

$$y > -\frac{1}{2}x + \frac{1}{2}$$

1pt: description

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Continue problem 5 on page 13.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

if x = 0 is crit #, then rel. min when $f'' > 0$
rel. max when $f'' < 0$

2nd derivative test ... ☺

$\left.\frac{dy}{dx}\right|_{(0,1)} = 0 + 1 - 1$
$= 0$ ∴ (0, 1) crit. pt.

$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 0 + 1 - \frac{1}{2}$
$= \frac{1}{2}$
> 0, ∴ f has rel. min @ x = 0

1pt: justification

1pt: answer

$f'' > 0$, f conc. up, min ... ☺

(d) Find the values of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

dy/dx

$\frac{dy}{dx} = \frac{1}{2}x + y - 1$

$m = \frac{1}{2}x + (mx + b) - 1$

$m = \frac{1}{2}x + mx + b - 1$

☺ ... linear = linear

$\frac{1}{2} + m = 0$

$m = -\frac{1}{2}$

constant = constant ... ☺

$b - 1 = m$

$b - 1 = -\frac{1}{2}$

$b = \frac{1}{2}$

1pt: value for m

1pt: value for b

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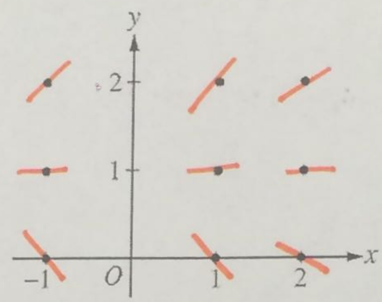
5. Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



1pt: zero slopes

1pt: other slopes

y\x	-1	1	2
2	1	1	1/4
1	0	0	0
0	-1	-1	-1/4

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Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

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(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

y's w/ dy, x's w/ dx

$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x^2}\,dx$$

1pt: separate variables

$u = y-1$

$du = dy$

$$\int \frac{1}{u}\,du = \int x^{-2}\,dx$$

$$\ln|u| = -x^{-1} + C$$

2pts: antiderive

$$\ln|y-1| = -\frac{1}{x} + C$$

1pt: "+C"

use $f(2)=0$

1pt: use initial condition

$$\ln|0-1| = -\frac{1}{2} + C$$

$$0 = -\frac{1}{2} + C$$

$$\frac{1}{2} = C$$

$$\ln|y-1| = -\frac{1}{x} + \frac{1}{2}$$

$$|y-1| = e^{-\frac{1}{x}+\frac{1}{2}}$$

$$y-1 = \pm e^{-\frac{1}{x}+\frac{1}{2}}$$

$$y-1 = -e^{-\frac{1}{x}+\frac{1}{2}}$$

$$\boxed{y = -e^{-\frac{1}{x}+\frac{1}{2}} + 1}$$

$y-1<0$

$0-1<0$

$-1<0$

keep "−"

1pt: solve for y

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to\infty}\left(-e^{-\frac{1}{x}+\frac{1}{2}} + 1\right)$$

$$= -e^{\frac{1}{2}} + 1$$

or $-\sqrt{e} + 1$

1pt: limit

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