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Lagrange and Alternating Series Error Bound Practice

1. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$f(x) \approx T_3(x) = -3 + 5(x-2) + \frac{3}{2!}(x-2)^2 - \frac{8}{3!}(x-2)^3$$

$$= -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{4}{3}(x-2)^3$$

$$f(1.5) \approx -3 + 5(1.5-2) + \frac{3}{2}(1.5-2)^2 - \frac{4}{3}(1.5-2)^3$$

$$\approx -4.958$$

b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$\therefore$, max of $|f^{(4)}(x)|$ is 3

centred @ $x=2$

$$|R_3(x)| \le \frac{\max|f^{(4)}(x)|}{4!}|x-2|^5$$

$$|R_3(1.5)| \le \frac{3}{4!}|1.5-2|^5$$

$$\le 0.0078125$$

actual value for $f(1.5)$ has error $\le 0.0078125 = R$ from approx of $f(1.5)$

$$|f(1.5) - T_3(1.5)| \le R_3(1.5)$$

$$-R_3(1.5) \le f(1.5) - T_3(1.5) \le R_3(1.5)$$

$$T_3(1.5) - R_3(1.5) \le f(1.5) \le T_3(1.5) + R_3(1.5)$$

$$-4.958 - .0078125 \le f(1.5) \le -4.958 + .0078125$$

$$-4.966 \le f(1.5) \le -4.951$$

$\therefore$, $f(1.5) \neq -5$

2. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{5^n \cdot n}$. When $x = 3.1$, the series converges to a value S. Use the first two terms of the series to approximate S. Use the alternating series error bound to show that this approximation differs from S by less than $\frac{1}{300{,}000}$.

$x = 3.1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(.1)^n}{5^n \cdot n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(50)^n \cdot n}$

alternating series w/ individual terms decreasing in abs value to zero.

$$S \approx \frac{(-1)^2}{50^1 \cdot 1} + \frac{(-1)^3}{50^2 \cdot 2}$$

$$\approx \frac{1}{50} - \frac{1}{5000} = \frac{99}{5000}$$

By alt. series error bound,

$$\left|S - \frac{99}{5000}\right| < \left|\frac{(-1)^4}{(50)^3 3}\right| = \frac{1}{375{,}000}$$

$$< \frac{1}{300{,}000}$$

3. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$

$x = \frac{1}{10}$

centered @ $x=0$

$|R_3(x)| = |f(x) - P_3(x)|$

$|R_3(x)| \le \frac{\max|f^{(4)}(x)|}{4!}x^4$

$|f(\frac{1}{10}) - P_3(\frac{1}{10})| \le \frac{625}{4!}(\frac{1}{10})^4$

$\le \frac{1}{384}$

Since $\frac{1}{384} < \frac{1}{100}$

$|f(\frac{1}{10}) - P_3(\frac{1}{10})| < \frac{1}{100}$

$f'(x) = 5\cos(5x - \frac{\pi}{4})$

$f''(x) = -25\sin(5x - \frac{\pi}{4})$

$f'''(x) = -125\cos(5x - \frac{\pi}{4})$

$f^{(4)}(x) = 625\sin(5x - \frac{\pi}{4})$

max $f^{(4)}(x)$ on $[0, \frac{1}{10}]$ is 625 (amplitude)

4. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{nx^n}{2n^2+1}$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series. The first ten terms of the Maclaurin series for f are used to approximate $f(-1)$. Show that this approximation differs from $f(-1)$ by less than $\frac{1}{10}$.

$f(-1) = \sum_{n=1}^{\infty} \frac{n(-1)^n}{2n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{2n^2+1}$

alternating series w/ individual terms decrease in abs value to 0.

alt. series error bound.

$\left|f(-1) - \sum_{n=1}^{10} \frac{(-1)^n n}{2n^2+1}\right| < \left|\frac{(-1)^{11} \cdot 11}{2(11)^2+1}\right| = \frac{11}{243}$

$< \frac{1}{10}$

5. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

a) Use $T(x)$ to find an approximation for $f(0)$.

$f(0) \approx 7 - 9(0-2)^2 - 3(0-2)^3$

$\boxed{f(0) \approx -5}$

b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0,2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (a) to explain why $f(0)$ is negative.

centered @ $x=2$

max of $f^{(4)}(x)$ is 6

$|R_3(x)| \le \frac{\max|f^{(4)}(x)|}{4!}|x-2|^4$

$\le \frac{6}{4!}|0-2|^4$

≤ 4

R

$|f(0) - P_3(0)| \le 4$

$-4 + P_3(0) \le f(0) \le 4 + P_3(0)$

$-4 - 5 \le f(0) \le 4 - 5$

$-9 \le f(0) \le -1$ $\therefore$, $f(0)$ is negative