DATE: ____________

AP FRQ

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

6. The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$\frac{\cos x - 1}{x^2} = \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots}{x^2} = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots \frac{(-1)^n x^{2n-2}}{(2n)!} + \dots$$

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

$f' = 0$, crit # rel. max $f'' < 0$, rel. min $f'' > 0$, neither $f'' = 0$:)

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-2)^2 + \dots$

$f(x) = -\frac{1}{2} + \frac{x^2}{4!} + \dots$

$f'(0) = 0$ (no x-term) :)

so @ $x = 0$, f has crit #.

$\frac{f''(0)}{2!} = \frac{1}{4!}$ (coefficent of x^2 term) :)

$f''(0) = \frac{1}{12}$

Since $f'(0) = 0$ and $f''(0) > 0$, then f has rel. max @ $x = 0$.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

$$g(x) = 1 + \int_0^x f(t)\,dt$$
$$= 1 + \int_0^x \left(-\frac{1}{2} + \frac{t^2}{4!} - \frac{t^4}{6!} + \dots\right) dt$$
$$= 1 + \left(-\frac{1}{2}t + \frac{1}{3 \cdot 4!}t^3 - \frac{1}{5 \cdot 6!}t^5 + \dots\right)\Big|_0^x$$
$$g(x) \approx 1 - \frac{1}{2}x + \frac{1}{3 \cdot 4!}x^3 - \frac{1}{5 \cdot 6!}x^5$$

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

$\hookrightarrow |g(1) - P_3(1)| < \frac{1}{6!}$?

$$P_3(x) = 1 - \frac{1}{2}x + \frac{x^3}{3 \cdot 4!}$$
$$g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!}$$
$$\approx \frac{37}{72}$$

$$\left|g(1) - \frac{37}{72}\right| \le |1^{st} \text{ unused term}|$$
$$\le \left|-\frac{1}{5 \cdot 6!}\right|$$
$$\le \frac{1}{5 \cdot 6!} < \frac{1}{6!}$$

b/c alt. series w/ individual terms dec in abs value to zero (alt series error bound)