Lengths of Parametric Functions

parametric functions $\rightarrow$ $x = f(t)$, $y = g(t)$

recall:

Distance $= \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$= \sqrt{\frac{(\Delta x)^2}{(\Delta t)^2} + \frac{(\Delta y)^2}{(\Delta t)^2}} \; \Delta t$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

arc length in parametric form $= \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$

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Parametric Curves (Arc Length)

Example 1

Find the length of the parametric curve $x = t^{3/2}$ and $y = 2t - 1$ on $[0,8]$

$\frac{dx}{dt} = \frac{3}{2}t^{1/2}$ $\quad \frac{dy}{dt} = 2$

$$L = \int_0^8 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^8 \sqrt{\left(\frac{3}{2}t^{1/2}\right)^2 + 2^2}\, dt$$

$u = \frac{9}{4}t + 4$ $\quad u(0) = 4$ $\quad u(8) = 22$

$$= \int_0^8 \sqrt{\frac{9}{4}t + 4}\, dt$$

$\frac{du}{dt} = \frac{9}{4}$

$\frac{4}{9}du = dt$

$$= \frac{4}{9}\int_4^{22} u^{1/2}\, du$$

$$= \frac{4}{9}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^{22} = \frac{8}{27}\left(22^{3/2} - 4^{3/2}\right) = \boxed{\frac{8}{27}\left(22^{3/2} - 8\right)}$$

Example 2

A particle moves along a curve so that its position is $(x(t), y(t))$ where $x(t) = t^2 - 4t + 8$ and $\frac{dy}{dt} = te^{t-3} - 1$, where x and y are measured in meters and t is measured in seconds.

a) Find the speed of the particle at $t = 3$.
b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

a) speed $= |v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ $\quad x'(t) = 2t - 4$

$|v(3)| = \sqrt{(x'(3))^2 + (y'(3))^2}$

$= 2.828$ meters/sec

b) total distance $= \int_0^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$= \int_0^4 \sqrt{(2t-4)^2 + (te^{t-3} - 1)^2}\, dt$

$= 11.587$ meters