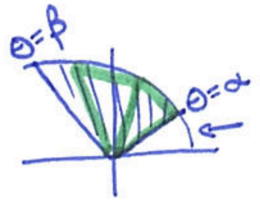
10.3 Area w/ Polar Functions

Area under curve $\rightsquigarrow$ Riemann Sums (rectangles)



$\leftarrow r = f(\theta)$ polar curve (rectangles not good)

sectors of circles is better.

$$\text{Area of Sector} = \frac{\theta}{2\pi}\overbrace{\pi r^2}^{\text{area of } \odot}$$

$$= \frac{1}{2}\theta r^2$$

$$= \frac{1}{2}(f(\theta))^2 \Delta\theta$$ $\rightarrow$ where $\Delta\theta$ is angle from α to β

$$\text{area of one sector} = \frac{1}{2}(f(\theta))^2 \Delta\theta$$

$$\text{sum of all sectors} = \lim_{\Delta\theta \to 0} \sum_{k=1}^{n} \frac{1}{2}(f(\theta_k))^2 \Delta\theta$$

$$\boxed{\text{Area "under" (Inside) Polar Curve} = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 \, d\theta}$$

Find the area enclosed by the curve $r = 1 - \cos\theta$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (1-\cos\theta)^2\, d\theta$$

$\alpha \to$ starting $\angle$

$\beta \to$ ending $\angle$

$$= \frac{1}{2}\int_{0}^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$$= \frac{1}{2}\int_{0}^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\right) d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$u = 2\theta$

$\frac{du}{d\theta} = 2$

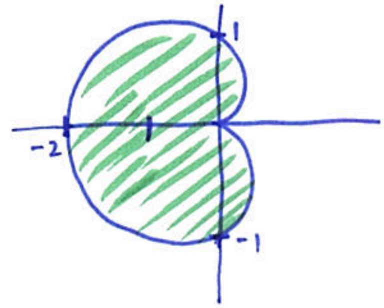
$\frac{du}{2} = d\theta$

$$= \frac{1}{2}\left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{2\pi}$$

$$= \frac{1}{2}\left(\frac{3}{2}\cdot 2\pi - 2\sin 2\pi + \frac{1}{4}\sin 4\pi - \left(\frac{3}{2}\cdot 0 - 2\sin 0 + \frac{1}{4}\sin 0\right)\right)$$

$$= \frac{1}{2}(3\pi - 0 + 0 - 0)$$

$$= \boxed{\frac{3\pi}{2}}$$



Find the area enclosed by the curve:

$r = 3\cos 2\theta$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (3\cos 2\theta)^2 \, d\theta \qquad \begin{aligned}\alpha &= 0\\ \beta &= 2\pi\end{aligned}$$

$$= \frac{1}{2}\int_{0}^{2\pi} (9\cos^2 2\theta)\, d\theta$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

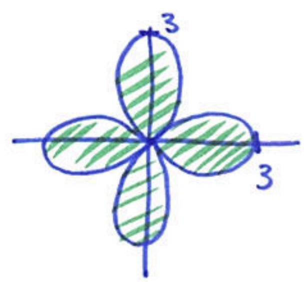
$$= \frac{9}{2}\int_{0}^{2\pi} \left(\tfrac{1}{2}\cos 4\theta + \tfrac{1}{2}\right) d\theta$$

$$= \frac{9}{2}\left(\frac{1}{8}\sin 4\theta + \frac{1}{2}\theta\right)\Big|_{0}^{2\pi}$$

$$= \frac{9}{2}\left(\frac{1}{8}\sin 8\pi + \frac{1}{2}\cdot 2\pi - \left(\frac{1}{8}\sin 0 + \frac{1}{2}\cdot 0\right)\right)$$

$$= \frac{9}{2}(0 + \pi - 0)$$

$$= \boxed{\frac{9}{2}\pi}$$



Find the area enclosed by the curve:

$r = 4\sin 3\theta$

$$A = \frac{1}{2}\int_0^{\pi} (4\sin 3\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (16\sin^2 3\theta)\, d\theta$$

$$= 8\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 6\theta\right) d\theta$$

$$= 8\left(\frac{1}{2}\theta - \frac{1}{12}\sin 6\theta\right)\Big|_0^{\pi}$$

$$= 8\left(\frac{1}{2}\cdot\pi - \frac{1}{12}\sin 6\pi - \left(\frac{1}{2}\cdot 0 - \frac{1}{12}\sin 0\right)\right)$$

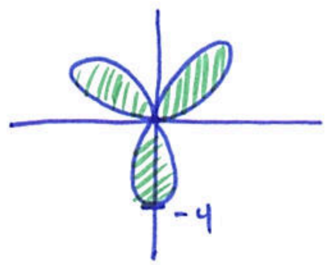
$$= 8\left(\frac{\pi}{2} - 0 - 0\right)$$

$$= \boxed{4\pi}$$

$1 - 2\sin^2\theta = \cos 2\theta$

$1 - \cos 2\theta = 2\sin^2\theta$

$\frac{1-\cos 2\theta}{2} = \sin^2\theta$

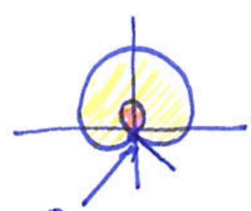


Set-up the integral to find the area inside the polar curve

$r = 2 + 4\sin\theta$

A = big loop − inner loop

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (2+4\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (2+4\sin\theta)^2\, d\theta$$



$r = 0$

$0 = 2 + 4\sin\theta$

$-2 = 4\sin\theta$

$-\frac{1}{2} = \sin\theta$

$\frac{7\pi}{6}, \frac{11\pi}{6} = \theta$