DATE:________________

10.3 Derivatives of Polar Functions

Recall from Pre-Calculus:

$x = r\cos\theta$ $r = f(\theta)$ $y = r\sin\theta$

$\therefore x = f(\theta)\cos\theta$ $y = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

where $\frac{dy}{d\theta} = \sin\theta\, f'(\theta) + f(\theta)\cos\theta$
$= \sin\theta \frac{dr}{d\theta} + r\cos\theta$

and $\frac{dx}{d\theta} = \cos\theta\, f'(\theta) + f(\theta)(-\sin\theta)$
$= \cos\theta \frac{dr}{d\theta} - r\sin\theta$

Examples:

1) Find the slope of the curve, $r = 3 + \sin\theta$, at $\left(3, \frac{\pi}{2}\right)$.

$x = (3+\sin\theta)\cos\theta$
$y = (3+\sin\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{\sin\theta(\cos\theta) + (3+\sin\theta)(\cos\theta)}{\cos\theta(\cos\theta) + (3+\sin\theta)(-\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{(3,\frac{\pi}{2})} = \frac{\sin\frac{\pi}{2}\cos\frac{\pi}{2} + (3+\sin\frac{\pi}{2})(\cos\frac{\pi}{2})}{\cos\frac{\pi}{2}\cos\frac{\pi}{2} + (3+\sin\frac{\pi}{2})(-\sin\frac{\pi}{2})}$$

$$= \frac{1(0) + (3+1)(0)}{0(0) + (3+1)(-1)}$$

$$= \frac{0}{-4} = \boxed{0}$$

2) Find the points where $r = 1 + \cos\theta$ has horizontal and vertical tangent lines.

$x = (1+\cos\theta)\cos\theta$
$y = (1+\cos\theta)\sin\theta$

$\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$

$\frac{dy}{d\theta} = 0$

$\frac{dy}{dx}$ undefined $\to \frac{\#}{0}$

$\frac{dy}{dx}$ undefined when $\frac{dx}{d\theta} = 0$

Horizontal Tangent line

$\frac{dy}{d\theta} = \sin\theta(-\sin\theta) + (1+\cos\theta)\cos\theta$
$0 = -\sin^2\theta + \cos\theta + \cos^2\theta$
$0 = -(1-\cos^2\theta) + \cos\theta + \cos^2\theta$
$0 = 2\cos^2\theta + \cos\theta - 1$
$0 = (2\cos\theta - 1)(\cos\theta + 1)$

$\cos\theta = \frac{1}{2}$ $\cos\theta = -1$
$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}}$ $\theta = \cancel{\pi}$

$\frac{dy}{dx} = \frac{0}{0}$ @ $\theta = \pi$

Vertical Tangent Line

$\frac{dx}{d\theta} = \cos\theta(-\sin\theta) + (1+\cos\theta)(-\sin\theta)$
$0 = -\cos\theta\sin\theta - \sin\theta - \cos\theta\sin\theta$
$0 = -2\cos\theta\sin\theta - \sin\theta$
$0 = -\sin\theta(2\cos\theta + 1)$

$-\sin\theta = 0$ $\cos\theta = -\frac{1}{2}$
$\boxed{\theta = 0, \cancel{\pi}}$ $\boxed{\theta = \frac{4\pi}{3}, \frac{5\pi}{3}}$

3) Find the slope of the line tangent to the curve, $r = \frac{2}{1+\sin\theta}$ at $\theta = -\frac{\pi}{4}$.

$$x = \frac{2}{1+\sin\theta}\cdot\cos\theta = \frac{2\cos\theta}{1+\sin\theta} \qquad y = \frac{2}{1+\sin\theta}\cdot\sin\theta \qquad y = \frac{2\sin\theta}{1+\sin\theta}$$

$$\frac{dy}{dx} = \frac{\dfrac{(1+\sin\theta)(2\cos\theta) - 2\sin\theta(\cos\theta)}{(1+\sin\theta)^2}}{\dfrac{(1+\sin\theta)(-2\sin\theta) - 2\cos\theta(\cos\theta)}{(1+\sin\theta)^2}}$$

$$= \frac{(1+\sin\theta)(2\cos\theta) - 2\sin\theta\cos\theta}{\cancel{(1+\sin\theta)^2}} \cdot \frac{\cancel{(1+\sin\theta)^2}}{(1+\sin\theta)(-2\sin\theta) - 2\cos\theta\cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=-\pi/4} = \frac{(1-\sqrt{2}/2)(2\cdot\sqrt{2}/2) - 2(-\sqrt{2}/2)(\sqrt{2}/2)}{(1-\sqrt{2}/2)(-2\cdot-\sqrt{2}/2) - 2(\sqrt{2}/2)(\sqrt{2}/2)} = \frac{\sqrt{2}-1+1}{\sqrt{2}-1-1} = \boxed{\frac{\sqrt{2}}{\sqrt{2}-2}}$$

4) Find the slope of the curve, $r = \theta - \cos\theta$, at $\theta = \frac{\pi}{2}$

$$x = (\theta - \cos\theta)\cos\theta$$

$$y = (\theta - \cos\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{\sin\theta(1+\sin\theta) + (\theta - \cos\theta)\cos\theta}{\cos\theta(1+\sin\theta) + (\theta-\cos\theta)(-\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{1(1+1) + (\pi/2 - 0)(0)}{0(1+1) + (\pi/2 - 0)(-1)}$$

$$= \frac{2}{-\pi/2}$$

$$= \boxed{-4/\pi}$$

5) Find the equation of the tangent line in terms of x and y for the curve, $r = 4\cos\theta$, at $\theta = \frac{3\pi}{4}$.

$$y - y_1 = m(x - x_1)$$

$$m \to \frac{dy}{dx}$$

$$x = 4\cos\theta\cos\theta$$

$$x(3\pi/4) = 4\cos 3\pi/4 \cos 3\pi/4 = 4(-\sqrt{2}/2)(-\sqrt{2}/2) = 2$$

$$\frac{dy}{dx} = \frac{\sin\theta(-4\sin\theta) + 4\cos\theta\cos\theta}{\cos\theta(-4\sin\theta) + 4\cos\theta(-\sin\theta)}$$

$$y = 4\cos\theta\sin\theta$$

$$y(3\pi/4) = 4\cos 3\pi/4 \sin 3\pi/4 = 4(-\sqrt{2}/2)(\sqrt{2}/2) = -2$$

$$\left.\frac{dy}{dx}\right|_{3\pi/4} = \frac{\sqrt{2}/2\left(-4\cdot\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)}{-\sqrt{2}/2\left(-4\cdot\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{-2+2}{2+2} = \frac{0}{4} = 0$$

$$\boxed{y + 2 = 0(x - 2)}$$