DATE: _______________

Taylor/Maclaurin Series and Taylor Polynomials

Example 1:

Write the third-degree Taylor Polynomial for $f(x) = e^{-x}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$f(x) = e^{-x} \approx 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!}$$

$$\boxed{e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}}$$

Example 2:

Write the Taylor Series, centered at $x = 0$, for $g(x) = \frac{x}{3x-1}$.

$\frac{a}{1-r}$ … ☺

$$= \frac{x}{-1(1-3x)} = -x\left(\frac{1}{1-3x}\right)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n$$

$$\frac{-x}{1-3x} = \sum_{n=0}^{\infty} -x(3x)^n$$ ← also ok

OR

$$\boxed{\frac{-x}{1-3x} = \sum_{n=0}^{\infty} -1 \cdot 3^n \cdot x^{n+1}}$$

Example 3:

Write the Taylor Series for $h(x) = \sin x$, centered at $x = \frac{3\pi}{2}$.

$h'(x) = \cos x$ $\quad h'''(x) = -\cos x$

$h''(x) = -\sin x$ $\quad h^{iv}(x) = \sin x$

$h^{v}(x) = \cos x$

↳ need to do Taylor Series work

$$h(x) = h\left(\tfrac{3\pi}{2}\right) + h'\left(\tfrac{3\pi}{2}\right)\left(x - \tfrac{3\pi}{2}\right) + \frac{h''\left(\frac{3\pi}{2}\right)}{2!}\left(x - \tfrac{3\pi}{2}\right)^2 + \frac{h'''\left(\frac{3\pi}{2}\right)}{3!}\left(x - \tfrac{3\pi}{2}\right)^3 + \frac{h^{iv}\left(\frac{3\pi}{2}\right)}{4!}\left(x - \tfrac{3\pi}{2}\right)^4 + \cdots + \frac{h^{(n)}\left(\frac{3\pi}{2}\right)}{n!}\left(x - \tfrac{3\pi}{2}\right)^n + \cdots$$

$$= -1 + 0 + \frac{1}{2!}\left(x - \tfrac{3\pi}{2}\right)^2 + 0 + \frac{-1}{4!}\left(x - \tfrac{3\pi}{2}\right)^4 + 0 + \cdots$$

$$\boxed{h(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\left(x - \frac{3\pi}{2}\right)^{2n}}{(2n)!}}$$

Example 4:

Write the Taylor Polynomial for $f(x) = 7x^2 - 6x + 1$ about $x = 2$.

↳ need to do Taylor Series work

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$ stop @ degree 2 … ☺

$$= 17 + 22(x-2) + \frac{14}{2!}(x-2)^2$$

$$\boxed{f(x) = 17 + 22(x-2) + 7(x-2)^2}$$

$f(2) = 7(2)^2 - 6(2) + 1$
$= 17$

$f'(x) = 14x - 6$

$f'(2) = 14(2) - 6 = 22$

$f''(x) = 14$

$f''(2) = 14$

Now you try...

1) Write the Taylor Polynomial of order 7 for $f(x) = \sin(3x)$.

degree 7

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

$$\sin 3x \approx 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}$$

$$\boxed{\sin 3x \approx 3x - \frac{9}{2}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7}$$

2) Write the Taylor Series, centered at $x = 0$, for $g(x) = x^6 e^{2x^3}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2x^3} = \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$$

$$x^6 e^{2x^3} = x^6 \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^6 \cdot 2^n \cdot x^{3n}}{n!} \Rightarrow \boxed{x^6 e^{2x^3} = \sum_{n=0}^{\infty} \frac{2^n x^{3n+6}}{n!}}$$

3) Write the Taylor Series for $h(x) = \frac{1}{x^2}$, centered at $x = -1$.

$h(x) = x^{-2}$ $\quad h(-1) = 1 \to 1!$

$h'(x) = -2x^{-3}$ $\quad h'(-1) = 2 \to 2!$

$h''(x) = 6x^{-4}$ $\quad h''(-1) = 6 \to 3!$

$h'''(x) = -24x^{-5}$ $\quad h'''(-1) = 24 \to 4!$

$$h(x) = h(-1) + h'(-1)(x+1) + \frac{h''(-1)}{2!}(x+1)^2 + \frac{h'''(-1)}{3!}(x+1)^3 + \dots \frac{h^{(n)}(-1)}{n!}(x+1)^n + \dots$$

$$= 1 + 2!(x+1) + \frac{3!}{2!}(x+1)^2 + \frac{4!}{3!}(x+1)^3 + \dots + \frac{(n+1)!}{n!}(x+1)^n + \dots$$

$$h(x) = \sum_{n=0}^{\infty} \frac{(n+1)!(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+1)n!}{n!}(x+1)^n \Rightarrow \boxed{h(x) = \sum_{n=0}^{\infty} (n+1)(x+1)^n}$$

4) Write the Taylor Polynomial for $f(x) = x^3 - 5x^2 - 1$ about $x = 3$.

degree 3

$f(3) = 3^3 - 5(3)^2 - 1 = -19$

$f'(x) = 3x^2 - 10x$ $\quad f'(3) = -3$

$f''(x) = 6x - 10$ $\quad f''(3) = 8$

$f'''(x) = 6$ $\quad f'''(3) = 6$

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$$

$$= -19 + -3(x-3) + \frac{8}{2!}(x-3)^2 + \frac{6}{3!}(x-3)^3$$

$$\boxed{f(x) = -19 + -3(x-3) + 4(x-3)^2 + (x-3)^3}$$