

### 3.6 Chain Rule

#### Chain Rule

$$\frac{d}{dx} (f(g(x))) = g'(x) \cdot f'(\underbrace{g(x)}_{\substack{\text{inside} \\ \text{derivative of inside}}})$$

↑ outside      ↑ inside  
derivative of outside

ex:  $y = (\underbrace{3x^2 + 4}_{\text{inside}})^5$  ← outside

$$g(x) = 3x^2 + 4$$

$$f(x) = x^5$$

$$y' = 6x \cdot 5(\underbrace{3x^2 + 4}_{\substack{\text{inside} \\ \text{deriv. inside}}})^4$$

↑ deriv. outside

$$y' = 30x(3x^2 + 4)^4$$

ex:  $f(x) = 3(\underbrace{4-x^2}_{\text{inside}})^4$  ← outside

$$f'(x) = -2x \cdot 12(4-x^2)^3$$

$$f'(x) = -24x(4-x^2)^3$$

ex:  $g(x) = \sqrt{x^2 - 2x + 1}$   
 $= (\underbrace{x^2 - 2x + 1}_{\text{inside}})^{\frac{1}{2}}$  ← outside

$$g'(x) = (2x - 2) \cdot \frac{1}{2}(x^2 - 2x + 1)^{-\frac{1}{2}}$$

$$= (x-1)(x^2 - 2x + 1)^{-\frac{1}{2}}$$

$$= (x-1)[(x-1)(x-1)]^{-\frac{1}{2}}$$

$$= (x-1)[(x-1)^2]^{-\frac{1}{2}}$$

$$\therefore \dots \cdot \dots \cdot \dots - (x-1)^0 = \boxed{1}$$

$$f(x) = \underbrace{x^2}_{f} \underbrace{(x-2)^4}_{g} \xrightarrow{\text{in}} \xleftarrow{\text{out}} \text{product}$$

$$\begin{aligned} f'(x) &= \underbrace{(x-2)^4}_{g} \cdot \underbrace{2x}_{f'} + \underbrace{x^2}_{f} \cdot \underbrace{1 \cdot 4(x-2)^3}_{g'} \\ &= 2x(x-2)^4 + 4x^2(x-2)^3 \quad \text{factor!} \\ &= 2x(x-2)^3 [x-2 + 2x] \end{aligned}$$

$$\boxed{f'(x) = 2x(x-2)^3(3x-2)}$$

$$\text{Ex: } f(t) = \sqrt{\frac{1}{t^2-2}}$$

$$= \underbrace{(t^2-2)}_{\text{in}}^{-1/2} \xleftarrow{\text{out}}$$

$$\begin{aligned} f'(t) &= 2t \cdot -\frac{1}{2}(t^2-2)^{-3/2} \\ &= -t(t^2-2)^{-3/2} \end{aligned}$$

$$\text{or} \quad = \frac{-t}{(t^2-2)^{+3/2}}$$

$$\text{or} \quad = \frac{-t}{\sqrt{(t^2-2)^3}}$$

ex:  $y = \sin(\pi x)$

$$y' = \pi \cdot \cos(\pi x)$$

$$\boxed{y' = \pi \cos \pi x}$$

ex:  $f(x) = \cos(\underline{\text{7}} - \underline{\text{5}}x)$

$$f'(x) = -5 \cdot -\sin(7-5x)$$

$$\boxed{f'(x) = 5 \sin(7-5x)}$$

ex:  $g(x) = \tan(\underline{\text{2}}x - \underline{\text{x}}^3)$

$$g'(x) = (2-3x^2) \sec^2(2x-x^3)$$

$$\boxed{g'(x) = (2-3x^2) \sec^2(2x-x^3)}$$

ex:  $f(x) = 5 \cos^2 x$

$$= 5 \underbrace{(\cos x)^2}_{\text{in}}^{\text{out}}$$

$$f'(x) = -\sin x \cdot 10(\cos x)'$$

$$\boxed{f'(x) = -10 \sin x \cos x}$$

ex:  $h(x) = \underbrace{5 \cos}_{\text{out}} \underbrace{(x^2)}_{\text{in}}$

$$h'(x) = 2x \cdot -5 \sin(x^2)$$

$$\boxed{h'(x) = -10x \sin x^2}$$

ex:  $g(x) = \sin(\cos \underline{\text{in}}^{\text{out}} 2x)$

$$g'(x) = 2 \cdot -\sin 2x \cdot \cos(\cos 2x)$$

$$\boxed{g'(x) = -2 \sin 2x \cos(\cos 2x)}$$

ex:  $h(x) = (\underbrace{1 + \cos 2x}_{\text{in}})^2 \leftarrow \text{out}$

$$h'(x) = (0 + 2 \cdot -\sin 2x) \cdot 2(1 + \cos 2x)$$

$$\boxed{h'(x) = -4 \sin 2x (1 + \cos 2x)}$$

ex:  $f(\theta) = \underbrace{\sec(2\theta)}_{f} \underbrace{\tan(2\theta)}_{g} \text{ product}$

$$f'(\theta) = \underbrace{\tan 2\theta}_{g} \cdot \underbrace{2 \cdot \sec 2\theta \tan 2\theta}_{f'} + \underbrace{\sec 2\theta}_{f} \cdot \underbrace{2 \cdot \sec^2(2\theta)}_{g'}$$

$$f'(\theta) = 2 \sec 2\theta \tan^2 2\theta + 2 \sec^3 2\theta$$

$$= 2 \sec 2\theta (\tan^2 2\theta + \sec^2 2\theta)$$

$$= 2 \sec 2\theta (\sec^2 2\theta - 1 + \sec^2 2\theta)$$

$$\boxed{f'(\theta) = 2 \sec 2\theta (2 \sec^2 2\theta - 1)}$$

$\tan^2 x + 1 = \sec^2 x$   
 $\tan^2 x = \sec^2 x - 1$

