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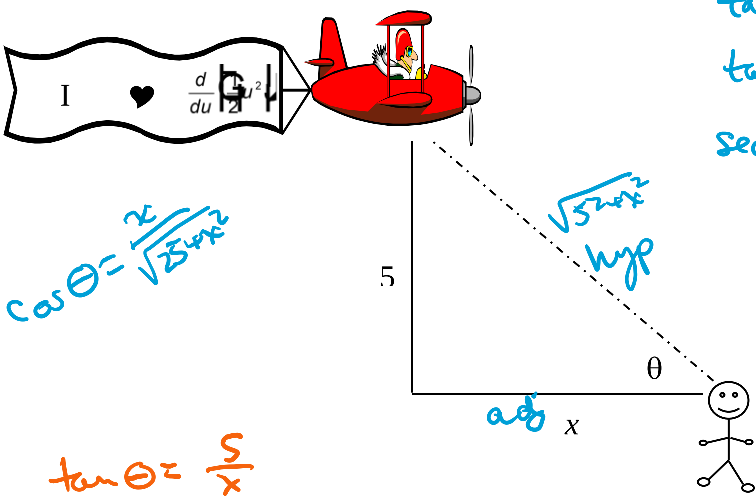
Derivatives of Inverse Trig Functions

You are looking up at a plane flying about 5 miles above the ground. As the plane move closer to you, the angle of your head/eyes changes. What is the rate at which that angle is changing with respect to x?



$\cos\theta = \frac{x}{\sqrt{25+x^2}}$

$\tan\theta = \frac{5}{x}$

$\tan\theta = 5x^{-1}$

$\sec^2\theta \frac{d\theta}{dx} = -5x^{-2}$

$\frac{d\theta}{dx} = \frac{-5}{x^2} \cdot \frac{1}{\sec^2\theta}$

$\frac{d\theta}{dx} = \frac{-5}{x^2} \cdot \cos^2\theta$

$= \frac{-5}{x^2}\left(\frac{x}{\sqrt{25+x^2}}\right)^2$

$= \frac{-5}{x^2} \cdot \frac{x^2}{(25+x^2)}$

$$\boxed{\frac{d\theta}{dx} = \frac{-5}{25+x^2}}$$

$\tan\theta = \frac{5}{x}$

$\theta = \tan^{-1}(5x^{-1})$

$\frac{d\theta}{dx} = -5x^{-2} \cdot \frac{1}{1+(5x^{-1})^2}$

$= \frac{-5}{x^2} \cdot \frac{1}{1+\frac{25}{x^2}}$

$$\boxed{\frac{d\theta}{dx} = \frac{-5}{x^2+25}}$$

same answer ☺

which is easier?

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Example 1:

Find $\frac{dy}{dx}$ for $y = \underbrace{x}_{f}\,\underbrace{\sin^{-1}(x)}_{g}$

$gf' + fg'$

$$\frac{dy}{dx} = \sin^{-1} x\,(1) + x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx} = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}}$$

Example 2:

Given position of an object as described by $x(t) = \frac{\tan^{-1} t}{t^2+3}$ where $t \geq 0$. Find the velocity of the object when $t = 1$.

↳ $x'(t)$

$x'(1)$

$$x'(t) = \frac{(t^2+3)\left(\frac{1}{1+t^2}\right) - (\tan^{-1} t)(2t)}{(t^2+3)^2}$$

$\tan^{-1} 1 \Rightarrow \tan\theta = 1?$

$\theta = \pi/4$

$$x'(1) = \frac{4 \cdot \frac{1}{2} - (\tan^{-1} 1)(2)}{(4)^2}$$

$$= \frac{2 - (\pi/4)(2)}{16}$$

$$= \left(2 - \frac{\pi}{2}\right)\left(\frac{1}{16}\right)$$

$$\boxed{v(1) = x'(1) = \frac{1}{8} - \frac{\pi}{32}}$$

What about Composite Functions? Chain Rule!

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} f(x)) = f'(x) \cdot \frac{1}{\sqrt{1-(f(x))^2}} \qquad \frac{d}{dx}(\cos^{-1} f(x)) = -f'(x) \cdot \frac{1}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}(\tan^{-1} f(x)) = f'(x) \cdot \frac{1}{1+(f(x))^2} \qquad \frac{d}{dx}(\cot^{-1} f(x)) = -f'(x) \cdot \frac{1}{1+(f(x))^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = f'(x) \cdot \frac{1}{|f(x)|\sqrt{(f(x))^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -f'(x) \cdot \frac{1}{|f(x)|\sqrt{(f(x))^2-1}}$$

Example 3:

Find $g'(x)$ for $g(x) = \underbrace{\cot^{-1}}_{\text{out}} \underbrace{\sqrt{x}}_{\text{in}}$

$$g'(x) = \frac{1}{2}x^{-\frac{1}{2}} \cdot -\frac{1}{1+(\sqrt{x})^2}$$

$$= -\frac{1}{2\sqrt{x}} \cdot \frac{1}{1+x}$$

$$\boxed{g'(x) = \frac{-1}{2\sqrt{x}(1+x)}}$$

Example 4:

Write the equation of the line tangent to the curve $h(x) = \cos^{-1}\left(\frac{x}{2} - 1\right)$ at $x = 3$.

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - \frac{\pi}{3} = -\frac{1}{\sqrt{3}}(x-3)}$$

$$h(3) = \cos^{-1}\left(\frac{3}{2} - 1\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$h(3) = \pi/3$$

$\cos\theta = \frac{1}{2}$?

$\theta = \pi/3$

$$h'(x) = \frac{1}{2} \cdot -\frac{1}{\sqrt{1-(\frac{x}{2}-1)^2}}$$

$$= -\frac{1}{2\sqrt{1-(\frac{x}{2}-1)^2}}$$

$$h'(3) = -\frac{1}{2\sqrt{1-(\frac{3}{2}-1)^2}}$$

$$= -\frac{1}{2\sqrt{1-(1/2)^2}}$$

$$= -\frac{1}{2\sqrt{1-1/4}}$$

$$= -\frac{1}{2\sqrt{3/4}}$$

$$= -\frac{1}{2 \cdot \frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$