8.4 Improper Integrals - integrals w/ infinite limits of integration.

If $f(x)$ cont on given interval limits, then

① $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

② $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

③ $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$ where c is a real #

$= \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$

Evaluate the integral

ex: $\int_1^{\infty} \frac{dx}{\sqrt{x}}$

$= \lim_{a\to\infty} \int_1^a x^{-1/2}\,dx$

$= \lim_{a\to\infty} (2x^{1/2})\Big|_1^a$

$= \lim_{a\to\infty} (2a^{1/2} - 2)$

$= \boxed{\infty}$ diverges

#1: $\int_{-\infty}^{-1} \frac{dx}{x^2}$

$= \lim_{a\to-\infty} \int_a^{-1} x^{-2}\,dx$

$= \lim_{a\to-\infty} (-x^{-1})\Big|_a^{-1}$

$= \lim_{a\to-\infty} (1 + \frac{1}{a})$

$= 1 + 0 = \boxed{1}$

#2: $\int_1^{\infty} \frac{5x+6}{x^2+2x}\,dx$

$= \lim_{a\to\infty} \int_1^a \frac{5x+6}{x^2+2x}\,dx$

$\frac{5x+6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$5x+6 = Ax + 2A + Bx$

$5 = A+B \quad 6 = 2A$

$5 = 3+B \quad 3 = A$

$2 = B$

$= \lim_{a\to\infty} \int_1^a \left(\frac{3}{x} + \frac{2}{x+2}\right) dx$

$= \lim_{a\to\infty} \left(3\ln|x| + 2\ln|x+2|\right)\Big|_1^a$

$= \lim_{a\to\infty} \left(3\ln a + 2\ln(a+2) - (3\ln 1 + 2\ln 3)\right)$

$= \lim_{a\to\infty} \left(\ln(a^3(a+2)^2) - \ln 3^2\right)$

$= \infty$ **diverges**

ex: $\int_0^\infty (x+1)e^{-x}\,dx$

$= \lim_{a\to\infty} \int_0^a (x+1)e^{-x}\,dx$

$u = x+1 \quad \int dv = \int e^{-x}\,dx$
$du = dx \quad v = -e^{-x}$

$= \lim_{a\to\infty} \left(-(x+1)e^{-x}\Big|_0^a - \int_0^a -e^{-x}\,dx\right)$

$= \lim_{a\to\infty} \left(-(a+1)e^{-a} + 1 + \int_0^a e^{-x}\,dx\right)$

$= \lim_{a\to\infty} \left(-ae^{-a} - e^{-a} + 1 - (e^{-x})\Big|_0^a\right)$

$= \lim_{a\to\infty} \left(-ae^{-a} - e^{-a} + 1 - e^{-a} + 1\right)$

$= \lim_{a\to\infty} \left(-ae^{-a} - 2e^{-a} + 2\right)$

$= \lim_{a\to\infty} \left(\frac{-a-2+2e^a}{e^a}\right) = \frac{\infty}{\infty}$ L'Hôpital

$= \lim_{a\to\infty} \left(\frac{-1+2e^a}{e^a}\right) = \frac{\infty}{\infty}$ L'Hôpital

$= \lim_{a\to\infty} \frac{2e^a}{e^a} = \lim_{a\to\infty} 2 = \boxed{2}$ converges