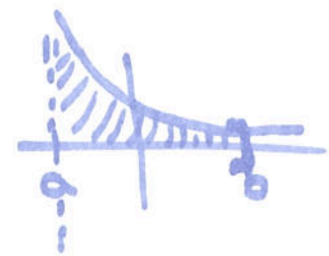
Other Improper Integrals

- asymptote (infinite limit @ point) within the interval of integration

① If $f(x)$ cont on $(a, b]$

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$



② If $f(x)$ cont on $[a, b)$,

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

③ If $f(x)$ cont on $[a, c) \cup (c, b]$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{d \to c^-} \int_a^d f(x)\,dx + \lim_{d \to c^+} \int_d^b f(x)\,dx$$

$$\text{ex: } \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to 1^-} \int_0^a \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{a\to 1^-} (\arcsin x)\Big|_0^a$$

$$= \lim_{a\to 1^-} \left(\sin^{-1}(a) - \sin^{-1}(0)\right)$$

$$= \lim_{a\to 1^-} \left(\sin^{-1}(a)\right)$$

$$= \boxed{\frac{\pi}{2}}$$

$$\text{ex: } \int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^4 x^{-1/2} e^{-x^{1/2}}\,dx$$

$$u = -x^{1/2}$$

$$\frac{du}{dx} = -\frac{1}{2}x^{-1/2}$$

$$= \lim_{a\to 0^+} \int_{-a^{1/2}}^{-2} x^{-1/2} e^{u} \cdot -2x^{1/2}\,du \quad \left\{ \begin{array}{l} 2x^{1/2}\,du = dx \\ u(a) = -a^{1/2} \\ u(4) = -4^{1/2} = -2 \end{array} \right.$$

$$= \lim_{a\to 0^+} -2\int_{-a^{1/2}}^{-2} e^{u}\,du$$

$$= \lim_{a\to 0^+} \left(-2e^{u}\right)\Big|_{-a^{1/2}}^{-2}$$

$$= \lim_{a\to 0^+} \left(-2e^{-2} + 2e^{-a^{1/2}}\right)$$

$$= \boxed{-2e^{-2} + 2}$$