Integration by Substitution

used when you have a function inside another function (chain)

$$\int f(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\,dx}_{du} = F(\underbrace{g(x)}_{u}) + C$$

$$\int f(u)\,du = F(u) + C$$

Find the indefinite integral.

ex: $\int (x^2-9)^3 \cdot 2x\,dx$

$u = x^2 - 9$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int u^3 \cdot \cancel{2x} \frac{du}{\cancel{2x}}$

$= \int u^3\,du$

$= \frac{1}{4}u^4 + C$

$= \frac{1}{4}(x^2-9)^4 + C$

① Look for function inside another function and set = to u.

② derive u

③ solve for dx

④ plug u & dx back into ∫

⑤ simplify (should only have u's - one kind of variable)

⑥ integrate

⑦ rewrite answer w/ x

ex: $\int \sqrt[3]{1-2x^2} \cdot 4x\,dx$

$u = 1-2x^2$

$\frac{du}{dx} = -4x$

$du = -4x\,dx$

$\frac{du}{-4x} = dx$

$= \int \sqrt[3]{u} \cdot 4x \cdot \frac{du}{-4x}$

$= -\int \sqrt[3]{u}\,du$

$= -\int u^{1/3}\,du$

$= -\frac{3}{4}u^{4/3} + C = -\frac{3}{4}(1-2x^2)^{4/3} + C$

ex: $\int \frac{x^3+1}{\sqrt{4x+x^4}}\,dx$

$u = 4x + x^4$

$\frac{du}{dx} = 4 + 4x^3$

$du = (4+4x^3)\,dx$

$\frac{du}{4+4x^3} = dx$

$= \int \frac{x^3+1}{\sqrt{u}} \cdot \frac{du}{4+4x^3}$

$= \int \frac{\cancel{x^3+1}}{\sqrt{u}} \cdot \frac{du}{4(\cancel{1+x^3})}$

$= \frac{1}{4}\int u^{-1/2}\,du$

$= \frac{1}{4}\left(\frac{1}{1/2}u^{1/2}\right) + C$

$= \frac{1}{4}(2u^{1/2}) + C = \boxed{\frac{1}{2}(4x+x^4)^{1/2} + C}$

Ex: $\int \csc(5x)\cot(5x)\,dx$

$u = 5x$

$\frac{du}{dx} = 5$

$du = 5\,dx$

$\frac{du}{5} = dx$

$= \int \csc u \cot u \cdot \frac{du}{5}$

$= \frac{1}{5}\int \csc u \cot u \, du$

$= \frac{1}{5}(-\csc u) + C$

$= -\frac{1}{5}\csc(5x) + C$

Ex: $\int x \sin(x^2)\,dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \int x \cdot \sin u \cdot \frac{du}{2x}$

$= \frac{1}{2}\int \sin u \, du$

$= \frac{1}{2}(-\cos u) + C$

$= -\frac{1}{2}\cos(x^2) + C$

Ex: $\int \sin^2 x \cos x \, dx$

$= \int (\sin x)^2 \cos x \, dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$\frac{du}{\cos x} = dx$

$= \int u^2 \cdot \cos x \cdot \frac{du}{\cos x}$

$= \int u^2 \, du$

$= \frac{1}{3} u^3 + C = \frac{1}{3} (\sin x)^3 + C$

$\text{OR } \frac{1}{3} \sin^3 x + C$