6.5 Logistic Growth

Partial Fractions

* Divide if improper fraction (exp N ≥ exp D)
* Factor denominator
* Decompose the fraction.

ex: $\int \frac{x+2}{x^2-4x}\,dx$

$$\cancel{x(x-4)}\,\frac{x+2}{\cancel{x(x-4)}} = \frac{A}{\cancel{x}}\,\cancel{x(x-4)} + \frac{B}{\cancel{x-4}}\,\cancel{x(x-4)}$$

$$x+2 = Ax - 4A + Bx$$

$$x+2 = Ax + Bx - 4A$$

$$x+2 = (A+B)x - 4A$$

$1 = A+B \qquad 2 = -4A$

$-\frac{1}{2} = A$

$1 = -\frac{1}{2} + B$

$\frac{3}{2} = B$

$$= \int \left(\frac{-\frac{1}{2}}{x} + \frac{3/2}{x-4} \right) dx$$

$$= -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{x-4}\,dx$$

$$= -\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x-4| + C$$

$\int (x-4)^{-1}\,dx$

$u = x-4$

$\int u^{-1}\,du \qquad \frac{du}{dx} = 1$

$\ln|u| \qquad du = dx$

$$= \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x| + C$$

$$= \ln|x-4|^{3/2} - \ln|x|^{1/2} + C$$

$$= \ln\left(\frac{|x-4|^{3/2}}{|x|^{1/2}}\right) + C \quad \underline{\text{OR}} \quad \ln\left(\frac{\sqrt{|x-4|^3}}{\sqrt{|x|}}\right) + C$$

$\underline{\text{ex}}$: $$\int \frac{x^2+2}{x^2-4x}\,dx$$

$$\begin{array}{r} 1 \\ x^2-4x\,\overline{\big)\, x^2 \quad +2} \\ -(x^2-4x) \\ \hline 4x+2 \end{array}$$

$4x+2$ (remainder)

$$= \int \left(1 + \frac{4x+2}{x^2-4x}\right)dx$$

$$= \int 1\,dx + \int \frac{4x+2}{x^2-4x}\,dx$$

$$\cancel{x(x-4)}\,\frac{4x+2}{\cancel{x(x-4)}} = \frac{A}{\cancel{x}}\,x(x-4) + \frac{B}{\cancel{x-4}}\,x(x-4)$$

$$4x+2 = Ax - 4A + Bx$$

$$4x+2 = Ax + Bx - 4A$$

$$4x+2 = (A+B)x - 4A$$

$$4 = A+B \qquad 2 = -4A$$

$$-\frac{1}{2} = A$$

$$4 = -\frac{1}{2} + B$$

$$\frac{9}{2} = B$$

$$= x + \int\left(\frac{-\frac{1}{2}}{x} + \frac{\frac{9}{2}}{x-4}\right)dx$$

$$= x + -\frac{1}{2}\int\frac{1}{x}\,dx + \frac{9}{2}\int\frac{1}{x-4}\,dx$$

$$= x - \frac{1}{2}\ln|x| + \frac{9}{2}\ln|x-4| + C$$

$$= x + \frac{9}{2}\ln|x-4| - \frac{1}{2}\ln|x| + C$$

$$= x + \ln\left(\frac{|x-4|^{9/2}}{|x|^{1/2}}\right) + C$$

ex: $\displaystyle\int_0^1 \frac{3}{2x^2+5x+2}\,dx$

$$\frac{3}{\cancel{(2x+1)(x+2)}}\,\cancel{(2x+1)(x+2)} = \frac{A}{\cancel{2x+1}}\,(\cancel{2x+1})(x+2) + \frac{B}{\cancel{x+2}}\,(2x+1)(\cancel{x+2})$$

$$3 = Ax + 2A + 2Bx + B$$

$$3 = (A+2B)x + 2A + B$$

$0 = A + 2B$ $\quad$ $3 = 2A + B$

$ \quad 0 = -2A - 4B$

$0 = A + 2(-1)$ $\quad$ $3 = -3B$

$0 = A - 2$ $\quad$ $-1 = B$

$2 = A$

$$\int_0^1 \left(\frac{2}{2x+1} + \frac{-1}{x+2}\right)dx$$

$$= 2\int_0^1 \frac{1}{2x+1}\,dx - 1\int_0^1 \frac{1}{x+2}\,dx$$

$$= \left[2\left(\tfrac{1}{2}\ln|2x+1|\right) - \ln|x+2|\right]\Big|_0^1$$

$$= \ln\left|\frac{2x+1}{x+2}\right|\Big|_0^1$$

$$= \ln\left|\frac{3}{3}\right| - \ln\left|\frac{1}{2}\right| = \ln 1 - (\ln 1 - \ln 2) = 0 - (0 - \ln 2) = \boxed{\ln 2}$$

ex: $\displaystyle\int \frac{5-x}{2x^2+x-1}\,dx$

$$\frac{5-x}{\cancel{(2x-1)(x+1)}}\,\cancel{(2x-1)(x+1)} = \frac{A}{\cancel{2x-1}}\,(\cancel{2x-1})(x+1) + \frac{B}{\cancel{x+1}}\,(2x-1)(\cancel{x+1})$$

$$5-x = Ax + A + 2Bx - B$$

$$5-x = (A+2B)x + A - B$$

$$-1 = A + 2B \qquad 5 = A - B$$

$$-5 = -A + B$$

$$-6 = 3B \qquad 5 = A-(-2)$$

$$-2 = B \qquad 5 = A+2$$

$$3 = A$$

$$\int\left(\frac{3}{2x-1} + \frac{-2}{x+1}\right)dx$$

$$= 3\int \frac{1}{2x-1}\,dx - 2\int\frac{1}{x+1}\,dx$$

$u = 2x-1$

$\frac{du}{dx} = 2$

$\frac{du}{2} = dx$

$\int \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\int u\,du$

$$= 3\left(\tfrac{1}{2}\ln|2x-1|\right) - 2\ln|x+1| + C$$

$$= \frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$$

$$= \ln\left(\frac{|2x-1|^{3/2}}{(x+1)^2}\right) + C \quad \text{OR} \quad \boxed{\ln\left(\frac{\sqrt{|2x-1|}^{\,3}}{(x+1)^2}\right) + C}$$