DATE: _____

## Lengths of Curves

Estimate the length of the curve $y = x^2$ on [1,3].

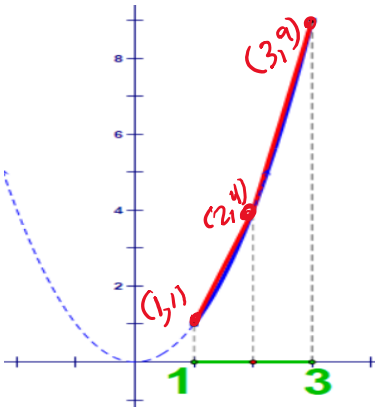
length of curve (distance) $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ ... (1)

$\approx \sqrt{(2-1)^2+(4-1)^2} + \sqrt{(3-2)^2+(9-4)^2}$

$\approx \sqrt{1+9} + \sqrt{1+25}$

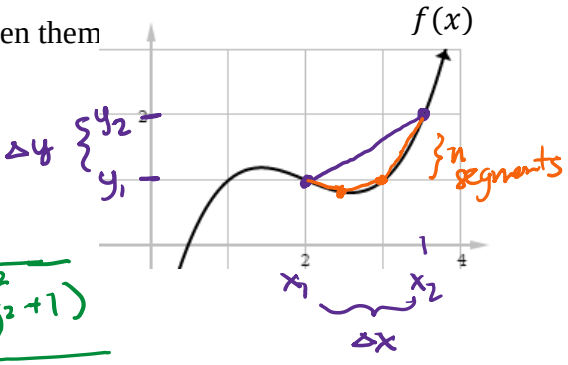
$\approx \sqrt{10} + \sqrt{26}$

$\approx 8.261$




## Finding the Length of a Curve

Given 2 points $(x_1, y_1)$ and $(x_2, y_2)$, find the distance between them




Distance $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(\Delta x)^2+(\Delta y)^2}$

$= \sqrt{(\Delta x)^2\left(1+\frac{(\Delta y)^2}{(\Delta x)^2}\right)}$ or $= \sqrt{(\Delta y)^2\left(\frac{(\Delta x)^2}{(\Delta y)^2}+1\right)}$

$= \Delta x\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}$ $= \Delta y\sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2+1}$

Sum of $n$ segments $= \sum_{k=1}^{n}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$ $\sum\sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2+1}\cdot\Delta y$

Sum of infinite # of segments

$= \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$ $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2+1}\cdot\Delta y$

Look familiar?

$= \lim_{\Delta x\to 0}\sum_{k=1}^{n}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$ $\lim_{\Delta x\to 0}\sum_{k=1}^{n}\sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2+1}\cdot\Delta y$

## Length of a Curve

**Length of a Curve = (Arc Length)** $\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

or $\int_c^d\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$

*Example 1:*

Find the length of the curve $y = x^2$ on $[1,3]$.

$\frac{dy}{dx} = 2x$ (a, b)

length curve $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

length of curve $= \int_1^3 \sqrt{1+(2x)^2}\,dx$

$= 8.268$

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*Example 2:*

Find the length of the curve $x = \sin y$ from $y = 0$ to $y = \frac{\pi}{2}$.

$\frac{dx}{dy} = \cos y$ (c, d)

length curve $= \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$

length of curve $= \int_0^{\pi/2} \sqrt{(\cos y)^2+1}\,dy$

$= 1.910$

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*Example 3:*

Find the length of the curve $y = \frac{1}{3}(x^2+2)^{3/2}$ on $[0,3]$.

(a, b)

length curve $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$\frac{dy}{dx} = 2x \cdot \frac{1}{3} \cdot \frac{3}{2}(x^2+2)^{1/2}$

$= x(x^2+2)^{1/2}$

length curve $= \int_0^3 \sqrt{1+(x(x^2+2)^{1/2})^2}\,dx$

$= \int_0^3 \sqrt{1+x^2(x^2+2)}\,dx$

$= \int_0^3 \sqrt{1+x^4+2x^2}\,dx$

$= \int_0^3 \sqrt{x^4+2x^2+1}\,dx$

$= \int_0^3 \sqrt{(x^2+1)(x^2+1)}\,dx$

$= \int_0^3 \sqrt{(x^2+1)^2}\,dx$

$= \int_0^3 (x^2+1)\,dx$

$= \left(\frac{1}{3}x^3 + x\right)\Big|_0^3$

$= 9 + 3 - 0 = \boxed{12}$

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*Example 4:*

Find the length of the curve $y = 2x^{1/5}$ from $(-32,-4)$ to $(32,4)$

(a, c) (b, d)

length curve $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

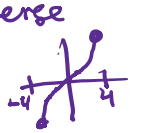
$\frac{dy}{dx} = \frac{2}{5}x^{-4/5}$

length curve $= \int_{-32}^{32} \sqrt{1+\left(\frac{2}{5}x^{-4/5}\right)^2}\,dx$

uh oh... $\frac{dy}{dx}$ DNE @ $x=0$ (which is in interval $(-32,32)$)

vertical tangent line @ $x=0$

use inverse



then, no DNE anymore

length curve $= \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$

$y = 2x^{1/5}$

$\frac{1}{2}y = x^{1/5}$

$\frac{1}{32}y^5 = x$

$\frac{dx}{dy} = \frac{5}{32}y^4$

length of curve $= \int_{-4}^{4} \sqrt{\left(\frac{5}{32}y^4\right)^2+1}\,dy$

$= \boxed{67.172}$