DATE: _____

Volume Using Cross Sections

Example 1:

Find the volume of a solid between $y = x^2 - x + 1$ and $y = x + 1$ whose cross sections perpendicular to the x-axis are squares.

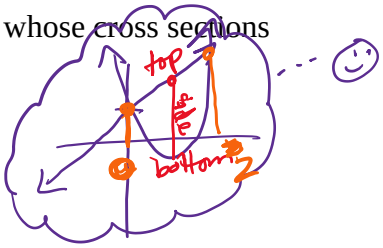
bottom: $x^2 - x + 1$ top: $x + 1$

$x^2 - x + 1 = x + 1$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, x = 2$



① square

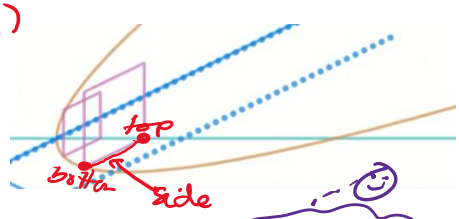
② Area of a square

$\text{Area} = (\text{side})^2$

$= (-x^2+2x)^2$

$\text{side} = x+1-(x^2-x+1)$

$= -x^2+2x$



$(-x^2+2x)^2 = (x^2+2x)(-x^2+2x)$

$= x^4 - 2x^3 - 2x^3 + 4x^2$

③ $\text{Volume} = \int_0^2 (-x^2+2x)^2\,dx$

$= \int_0^2 (x^4 - 4x^3 + 4x^2)\,dx$

$= \left(\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3\right)\Big|_0^2$

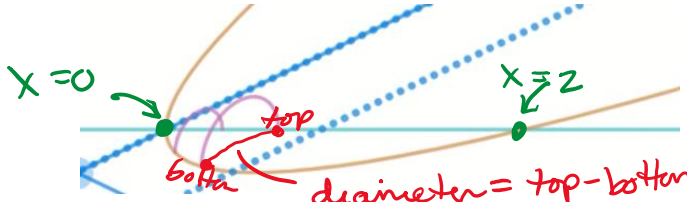
$= \frac{32}{5} - 16 + \frac{32}{3} - 0 = \boxed{16/3}$

Example 2:

Find the volume of a solid between $y = x^2 - x + 1$ and $y = x + 1$ whose cross sections perpendicular to the x-axis are semicircles.

bottom: $x^2 - x + 1$ top: $x + 1$

① semicircles



② Area of semicircle

$\text{Area} = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi\left(\frac{1}{2}(-x^2+2x)\right)^2$

$= \frac{1}{2}\pi\left(\frac{1}{4}\right)(-x^2+2x)^2$

$= \frac{1}{8}\pi(-x^2+2x)^2$

$\text{radius} = \frac{1}{2}(x+1-(x^2-x+1))$

$= \frac{1}{2}(-x^2+2x)$

$\text{diameter} = \text{top} - \text{bottom}$

$\text{radius} = \frac{1}{2}(\text{diameter})$

$= \frac{1}{2}(\text{top} - \text{bottom})$

③ $\text{Volume} = \int_0^2 \frac{1}{8}\pi(-x^2+2x)^2\,dx$

$= \frac{1}{8}\pi\int_0^2 (-x^2+2x)^2\,dx$

$= \frac{1}{8}\pi\left(\frac{16}{3}\right) = \boxed{\frac{2}{3}\pi}$

Example 3:

Find the volume of a solid between $y = x^2 - x + 1$ and $y = x + 1$ whose cross sections perpendicular to the x-axis are rectangles whose height is thrice its width.

3 times

① rectangles

② Area of Rectangle

Area = (base)(height)

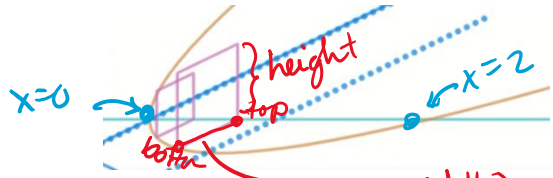
$= (-x^2+2x)(3(-x^2+2x))$

$= 3(-x^2+2x)^2$

base $= x+1-(x^2-x+1)$

$= -x^2+2x$

height $= 3(-x^2+2x)$



base (or width)
= top − bottom

height = 3(width)
= 3(top − bottom)

③ Volume $= \int_0^2 3(-x^2+2x)^2\,dx$

$= 3\int_0^2 (-x^2+2x)^2\,dx$

$= 3\left(\frac{16}{3}\right)$

$= \boxed{16}$

Example 4:

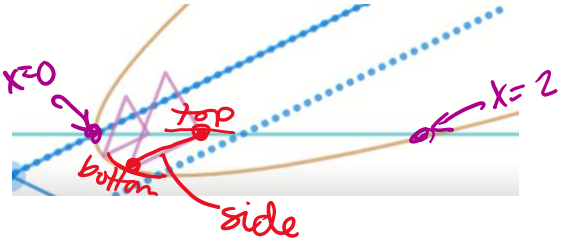
Find the volume of a solid between $y = x^2 - x + 1$ and $y = x + 1$ whose cross sections perpendicular to the x-axis are equilateral triangles.

① equilateral △s

② Area of equilateral △

Area $= \frac{\sqrt{3}}{4}(\text{side})^2$

$= \frac{\sqrt{3}}{4}(-x^2+2x)^2$



side = top − bottom

side $= x+1-(x^2-x+1)$

$= -x^2+2x$

③ Volume $= \int_0^2 \frac{\sqrt{3}}{4}(-x^2+2x)^2\,dx$

$= \frac{\sqrt{3}}{4}\int_0^2 (-x^2+2x)^2\,dx$

$= \frac{\sqrt{3}}{4}\left(\frac{16}{3}\right) = \boxed{\frac{4\sqrt{3}}{3}}$