8.2 L'Hôpital's Rule

If $f(a) = g(a) = 0$, $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$,

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

This means: If $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} = \frac{0}{0}$,

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

ex: $$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{3^2 - 9}{3 - 3} = \frac{0}{0}$$

so by L'Hôpital

$$\lim_{x \to 3} \frac{2x}{1} = \frac{2(3)}{1} = \boxed{6}$$

ex: $$\lim_{x \to 2} \frac{\sqrt{x+2} - 2}{x - 2} = \frac{\sqrt{2+2} - 2 = \sqrt{4} - 2}{2 - 2} = \frac{0}{0}$$

so by L'Hôpital

$$\lim_{x \to 2} \frac{(x+2)^{1/2} - 2}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{2}(x+2)^{-1/2}}{1}$$

$$= \frac{1}{2}(4)^{-1/2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$$

ex: $\lim\limits_{x\to\infty} \dfrac{x-3}{x^2-9} = \dfrac{\infty}{\infty}$

$\lim\limits_{x\to\infty} \dfrac{1}{2x}$

$= \dfrac{1}{\infty} = \boxed{0}$

L'Hôpital's Rule also applies to $\dfrac{\infty}{\infty}$ ...

ex: $\lim\limits_{x\to\infty} \dfrac{x^2-9}{x-3} = \dfrac{\infty}{\infty}$

$\lim\limits_{x\to\infty} \dfrac{2x}{1}$

$= \boxed{\infty}$

L'Hôpital w/ $\infty \cdot 0$, $\infty - \infty$, 1^∞, 0^0, ∞^0

Get functions of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then you can use L'Hôpital's Rule

ex: $\lim\limits_{x \to 0} \left(\frac{1}{x} - \cot x \right) = \frac{1}{0} - \cot 0 = \infty - \infty$

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right)$$

$$\lim_{x \to 0} \left(\frac{\sin x - x\cos x}{x \sin x} \right) = \frac{0-0}{0} = \frac{0}{0}$$

* by L'Hôpital

$$\lim_{x \to 0} \left(\frac{\cos x - (\cos x \cdot 1 + x(-\sin x))}{\sin x(1) + x(\cos x)} \right)$$

$$\lim_{x \to 0} \frac{x \sin x}{\sin x + x\cos x} = \frac{0}{0+0} = \frac{0}{0}$$

* L'Hôpital again....

$$\lim_{x \to 0} \frac{\sin x(1) + x(\cos x)}{\cos x + \cos x \cdot 1 + x(-\sin x)}$$

$$\lim_{x \to 0} \frac{\sin x + x\cos x}{2\cos x - x\sin x} = \frac{0+0}{2-0} = \frac{0}{2} = \boxed{0}$$

ex: $\lim_{x\to 0} (\cot x)(x^2+5x) = \infty \cdot 0$:(

$\lim_{x\to 0} \left(\frac{\cos x}{\sin x}\right)(x^2+5x) = \frac{0}{0}$:)

* by L'Hôpital

$$\lim_{x\to 0} \frac{(x^2+5x)(-\sin x) + (\cos x)(2x+5)}{\cos x}$$

$$= \frac{0 + 1(5)}{1}$$

$$= \boxed{5}$$

ex: $\lim_{x\to 0^+} (5x)^{3x} = 0^0$:(

let $f(x) = (5x)^{3x}$

$\ln f(x) = \ln(5x)^{3x}$

$\ln f(x) = 3x \ln(5x)$

$\lim_{x\to 0^+} 3x \ln(5x) = 0 \cdot \infty$:(

$\lim_{x\to 0^+} \frac{\ln(5x)}{1/3x} = \frac{\infty}{\infty}$:)

← by L'Hôpital

$\lim_{x\to 0^+} \frac{\frac{1}{5x} \cdot 5}{-\frac{1}{3}x^{-2}}$

$\lim_{x\to 0^+} \frac{1}{x} \cdot \frac{-3x^2}{1}$

$\lim_{x\to 0^+} -3x = 0$

so, $e^{\ln f(x)} = e^0 = \boxed{1}$

ex: $\lim_{x \to 0^+} \sqrt{x} \ln x = 0 \cdot \infty$ ☹

$\lim_{x \to 0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}} = \dfrac{\infty}{\infty}$ ☺

* by L'Hôpital's,

$\lim_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$

$\lim_{x \to 0^+} \frac{1}{x} \cdot -2x^{3/2}$

$\lim_{x \to 0^+} -2x^{1/2} = \boxed{0}$

ex: $\lim_{x \to 0} (1-x)^{1/x} = 1^{\infty}$ ☹

$f(x) = (1-x)^{1/x}$

$\ln f(x) = \ln(1-x)^{1/x}$

$= \frac{1}{x} \ln(1-x)$

$\lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \frac{1}{x} \ln(1-x) = \infty \cdot 0$ ☹

$= \lim_{x \to 0} \dfrac{\ln(1-x)}{x} = \dfrac{0}{0}$ ☺

* by L'Hôpital

$= \lim_{x \to 0} \dfrac{\frac{1}{1-x} \cdot -1}{1}$

$= \lim_{x \to 0} -\frac{1}{1-x} = -1$

$e^{\lim_{x \to 0} \ln f(x)} = e^{-1} = \boxed{\frac{1}{e}}$

ex: $\lim_{x\to 0}(\sin x)^x = 0^0$

$$f(x) = (\sin x)^x$$

$$\ln f(x) = \ln(\sin x)^x$$

$$= x\ln(\sin x)$$

$$\lim_{x\to 0}\ln f(x) = \lim_{x\to 0} x\ln(\sin x) = 0\cdot\infty$$

$$= \lim_{x\to 0}\frac{\ln(\sin x)}{\frac{1}{x}} = \frac{\infty}{\infty}$$

* by L'Hôpital,

$$= \lim_{x\to 0}\frac{\frac{1}{\sin x}\cdot\cos x}{-x^{-2}}$$

$$= \lim_{x\to 0}\frac{\cos x}{\sin x}\cdot x^2 = \frac{0}{0}$$

* L'Hôpital again...

$$= \lim_{x\to 0}\frac{x^2(-\sin x) + \cos x(2x)}{\cos x}$$

$$= \frac{0+0}{1} = 0$$

$$e^{\lim_{x\to 0}\ln f(x)} = e^0 = \boxed{1}$$