8.4 Improper Integrals - integrals w/ infinite limits of integration.

If $f(x)$ cont on given interval limits, then

① $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

② $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

③ $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$ where c is a real #

$= \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$

Evaluate the integral

ex: $\int_1^{\infty} \frac{dx}{\sqrt{x}}$

$= \lim_{a\to\infty} \int_1^a x^{-1/2}\,dx$

$= \lim_{a\to\infty} \left(2x^{1/2}\right)\Big|_1^a$

$= \lim_{a\to\infty} \left(2a^{1/2} - 2\right)$

$= \boxed{\infty}$ diverges

ex: $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2}$

$$= \lim_{a\to-\infty} \int_a^{-1} x^{-2}\,dx$$

$$= \lim_{a\to-\infty} \left(-x^{-1}\right)\Big|_a^{-1}$$

$$= \lim_{a\to-\infty} \left(1+\frac{1}{a}\right)$$

$$= 1+0 = \boxed{1}$$

ex: $\displaystyle\int_1^{\infty} \frac{5x+6}{x^2+2x}\,dx$

$$= \lim_{a\to\infty} \int_1^{a} \frac{5x+6}{x^2+2x}\,dx$$

$$\frac{5x+6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$5x+6 = Ax+2A+Bx$$

$5 = A+B \quad 6 = 2A$

$5 = 3+B \quad 3 = A$

$2 = B$

$$= \lim_{a\to\infty} \int_1^{a} \left(\frac{3}{x}+\frac{2}{x+2}\right)dx$$

$$= \lim_{a\to\infty} \left(3\ln|x| + 2\ln|x+2|\right)\Big|_1^{a}$$

$$= \lim_{a\to\infty} \left(3\ln a + 2\ln(a+2) - (3\ln 1 + 2\ln 3)\right)$$

$$= \lim_{a\to\infty} \left(\ln\left(a^3(a+2)^2\right) - \ln 3^2\right)$$

$$= \infty \quad \boxed{\text{diverges}}$$

ex: $\int_0^\infty (x+1)e^{-x}\,dx$

$= \lim_{a\to\infty} \int_0^a (x+1)e^{-x}\,dx$

$u = x+1 \quad \int dv = \int e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$= \lim_{a\to\infty} \left(-(x+1)e^{-x}\Big|_0^a - \int_0^a -e^{-x}\,dx \right)$

$= \lim_{a\to\infty} \left(-(a+1)e^{-a} + 1 + \int_0^a e^{-x}\,dx \right)$

$= \lim_{a\to\infty} \left(-ae^{-a} - e^{-a} + 1 - (e^{-x})\Big|_0^a \right)$

$= \lim_{a\to\infty} \left(-ae^{-a} - e^{-a} + 1 - e^{-a} + 1 \right)$

$= \lim_{a\to\infty} \left(-ae^{-a} - 2e^{-a} + 2 \right)$

$= \lim_{a\to\infty} \left(\frac{-a - 2 + 2e^{a}}{e^{a}} \right) = \frac{\infty}{\infty}$ L'Hôpital

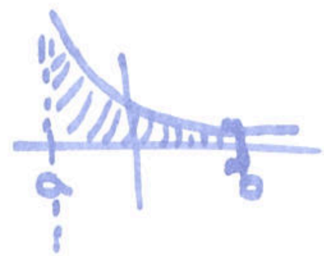
$= \lim_{a\to\infty} \left(\frac{-1 + 2e^{a}}{e^{a}} \right) = \frac{\infty}{\infty}$ L'Hôpital

$= \lim_{a\to\infty} \frac{2e^{a}}{e^{a}} = \lim_{a\to\infty} 2 = \boxed{2}$ converges

Other Improper Integrals - asymptote (infinite limit @ point) within the interval of integration

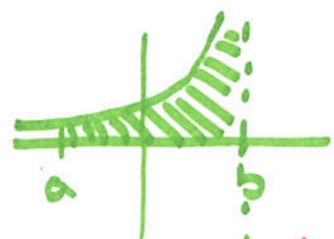
① If $f(x)$ cont on $(a, b]$

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$



② If $f(x)$ cont on $[a, b)$,

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$



③ If $f(x)$ cont on $[a, c) \cup (c, b]$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{d \to c^-} \int_a^d f(x)\,dx + \lim_{d \to c^+} \int_d^b f(x)\,dx$$

ex: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to 1^-} \int_0^a \frac{1}{\sqrt{1-x^2}}\,dx$

$= \lim_{a\to 1^-} (\arcsin x)\Big|_0^a$

$= \lim_{a\to 1^-} \left(\sin^{-1}(a) - \sin^{-1}(0)\right)$

$= \lim_{a\to 1^-} \left(\sin^{-1}(a)\right)$

$= \boxed{\frac{\pi}{2}}$

ex: $\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^4 x^{-1/2} e^{-x^{1/2}}\,dx$

$u = -x^{1/2}$

$\frac{du}{dx} = -\frac{1}{2}x^{-1/2}$

$= \lim_{a\to 0^+} \int_{-a^{1/2}}^{-2} x^{-1/2} e^{u} \cdot -2x^{1/2}\,du$ $\Bigg\}$ $2x^{1/2}du = dx$

$u(a) = -a^{1/2}$

$u(4) = -4^{1/2} = -2$

$= \lim_{a\to 0^+} -2\int_{-a^{1/2}}^{-2} e^{u}\,du$

$= \lim_{a\to 0^+} \left(-2e^{u}\right)\Big|_{-a^{1/2}}^{-2}$

$= \lim_{a\to 0^+} \left(-2e^{-2} + 2e^{-a^{1/2}}\right)$

$= \boxed{-2e^{-2} + 2}$