DATE: _____

Power Series

Infinite Series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

Finding the sum of an infinite Series by finding the Partial Sums

If a partial sum converges to S, then the series converges to S.

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = \sum_{k=1}^{\infty} a_k$$

Example: Does the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$ converge?

$S_1 = 1$
$S_2 = 1 + \frac{1}{2} = 1.5$
$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75$
$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$
$\vdots$
$S_n = \sum_{k=1}^{\infty} \frac{1}{2^k} = 2$

$S_1 = 2 - 1 = 1$
$S_2 = 2 - \frac{1}{2} = 1.5$
$S_3 = 2 - \frac{1}{4} = 2 - \frac{1}{2^2} = 1.75$
$S_3 = 2 - \frac{1}{8} = 2 - \frac{1}{2^3} = 1.875$
$\vdots$
$S_n = 2 - \frac{1}{2^n}$
$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2 - 0 = 2$

✱If the infinite series converges, then $\lim_{k\to\infty} a_k = 0$.

(The converse is not necessarily true, but the contrapositive is true.)

Converse:

If $\lim_{k\to\infty} a_k = 0$, then the infinite series converges. (not necessarily true)

$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} + \cdots$ $\lim_{k\to\infty} \frac{1}{k} = 0$, but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

Contrapositive:

✱ If $\lim_{k\to\infty} a_k \neq 0$, then the infinite series diverges. (is true)

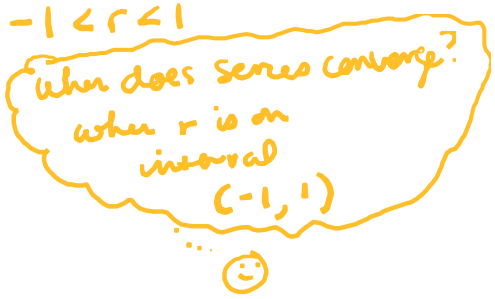
n^th^ term test

$\sum_{k=1}^{\infty} 2^k = 2 + 2^2 + 2^4 + \cdots + 2^k + \cdots$ $\lim_{k\to\infty} 2^k \neq 0$, $\therefore \sum_{k=1}^{\infty} 2^k$ diverges

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} ar^n$$

converges to : $\frac{a}{1-r}$, if $|r| < 1$ ← interval of convergence: $-1 < r < 1$



diverges if: $|r| \geq 1$

Power Series

Centered at $x = 0$,

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

Example:

$$\sum_{n=0}^{\infty} \frac{x^n}{n} = \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

Centered at $x = a$,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots + c_n (x-a)^n + \cdots$$

Example:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n} = (x-2)^1 + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \cdots \frac{(x-2)^n}{n} + \cdots$$

Example 1:
Write the Geometric Series below as a function.

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + \cdots + x^n + \cdots$$
$$= 1 + x + x^2 + \cdots + x^n + \cdots$$
$$= \frac{1}{1-x} \quad \text{when } |x|<1$$

$\frac{a}{1-r}$

So, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on the interval of convergence $(-1,1)$

Example 2:
Express the function as a power series and find the interval of convergence.

$f(x) = \frac{1}{1+x}$

$|r|<1$

$\frac{a}{1-r}$

$$= \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots + (-x)^n + \cdots$$
$$= \sum_{n=0}^{\infty} (-x)^n$$

So, $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$
$= \sum_{n=0}^{\infty} (-1)^n x^n$

$|-x|<1$
$|-1||x|<1$
$|x|<1$
$-1<x<1$

interval of convergence : $(-1,1)$

Example 3:
Express the function as a power series and find the interval of convergence.

$g(x) = \frac{1}{1+x^2}$

$|r|<1$

$\frac{a}{1-r}$

$$= \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + \cdots + (-x^2)^n + \cdots$$
$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^{2n})$

$|-x^2|<1$
$\sqrt{|x^2|}<\sqrt{1}$
$|x|<1$
$-1<x<1$

interval of convergence : $(-1,1)$

Example 4:
Express the function as a power series and find the interval of convergence.

$h(x) = \frac{1}{5+x}$

$\frac{a}{1-r}$ ..

$= \frac{1}{5\left(1+\frac{1}{5}x\right)}$

$= \frac{\frac{1}{5}}{1-\left(-\frac{1}{5}x\right)}$ ← a, ← r

$\frac{1}{5+x} = \sum_{n=0}^{\infty} \frac{1}{5}\left(-\frac{1}{5}x\right)^n$

$= \sum_{n=0}^{\infty} \frac{1}{5}(-1)^n\left(\frac{1}{5}\right)^n x^n$

$\frac{1}{5+x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^{n+1}}$

$|r|<1$

$\left|-\frac{x}{5}\right|<1$

$\left|\frac{x}{5}\right|<1$

$-1<\frac{x}{5}<1$

$-5<x<5$

interval of convergence: $(-5, 5)$

Example 5:
Express the function as a power series and find the interval of convergence.

$p(x) = \frac{x^5}{3-6x^3}$

$\frac{a}{1-r}$ ..

$= \frac{x^5}{3(1-2x^3)}$

$= \frac{\frac{x^5}{3}}{1-2x^3}$ ← a, ← r

$= \sum_{n=0}^{\infty} \frac{x^5}{3}\left(2x^3\right)^n$

$= \sum_{n=0}^{\infty} \frac{x^5}{3} \cdot 2^n x^{3n}$

$\frac{x^5}{3-6x^3} = \sum_{n=0}^{\infty} \frac{1}{3} \cdot 2^n x^{5+3n}$

$|r|<1$

$|2x^3|<1$

$\sqrt[3]{|x^3|}<\sqrt[3]{\frac{1}{2}}$

$|x|<\sqrt[3]{\frac{1}{2}}$

$-\sqrt[3]{\frac{1}{2}}<x<\sqrt[3]{\frac{1}{2}}$

interval of convergence: $\left(-\sqrt[3]{\frac{1}{2}}, \sqrt[3]{\frac{1}{2}}\right)$