Power Series

Recall the Power Series, centered at $x = a$:

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$

Power Series Theorems

If

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

is diff'able and converges on $\underbrace{(a-R, a+R)}_{\text{interval of convergence}}$,

then

① $f'(x) = 0 + c_1 + c_2 \cdot 2(x-a) + 3c_3(x-a)^2 + \ldots + nc_n(x-a)^{n-1} + \cdots$

or $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

AND

② $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + c_n\frac{(x-a)^{n+1}}{n+1} + \cdots$

or $= C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$

also,

$$\int_a^x f(t)\,dt = \left(c_0(t-a) + \frac{c_1(t-a)^2}{2} + \cdots + \frac{c_n(t-a)^{n+1}}{n+1} + \cdots\right)\Big|_a^x$$

$$= c_0(x-a) + \frac{c_1(x-a)^2}{2} + \cdots \frac{c_n(x-a)^{n+1}}{n+1} + \cdots - (0)$$

$$= c_0(x-a) + \frac{c_1(x-a)^2}{2} + \cdots + \frac{c_n(x-a)^{n+1}}{n+1}$$

or $= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

- If the series for f converges for all x, then so does the series for f' and the series for $\int f(x)\,dx$.

Express each function as a power series.

Example 1:
$\frac{1}{(1-x)^2}$

not of form $\frac{a}{1-r}$ ... but $\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left((1-x)^{-1}\right)$
$= -1(-1)(1-x)^{-2}$
$= \frac{1}{(1-x)^2}$

$$\frac{1}{1-x} = 1+x+x^2+\cdots+x^n+\cdots$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 1+2x+\cdots+nx^{n-1}+\cdots$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}}$$

Example 2:
$\ln(1-x)$

not of form $\frac{a}{1-r}$, but $\int \frac{1}{1-x}dx = -\ln|1-x|$
$-\int \frac{1}{1-x}dx = \ln|1-x|$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}dx = C+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$-\ln|1-x| = C+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

centered @ $x=0$

$$-\ln|1-0| = C+\sum_{n=0}^{\infty}\frac{0^{n+1}}{n+1}$$

$$0 = C$$

$$-\ln|1-x| = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$\ln|1-x| = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{-1\cdot x^{n+1}}{n+1}$$

OR $\sum_{n=1}^{\infty}\frac{-1x^n}{n}$

Example 3:
$\tan^{-1}x$

not of form $\frac{a}{1-r}$, but $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int\frac{1}{1+x^2}dx = C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1}x = C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

centered @ $x=0$

$$\tan^{-1}0 = C+\sum_{n=0}^{\infty}(-1)^n\frac{0^{2n+1}}{2n+1}$$

$$0 = C$$

$$\boxed{\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}}$$