DATE: ___________

9.2 Taylor Series

Recall the Power Series:

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$

for $x = a$, $f(a) = c_0$

so, $c_0 = f(a)$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1} + \cdots$

then, $f'(a) = c_1$

so, $c_1 = f'(a)$

$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots + (n-1)n\cdot c_n(x-a)^{n-2} + \cdots$

then, $f''(a) = 2c_2$

so, $c_2 = \frac{f''(a)}{2}$

$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)n\cdot c_n(x-a)^{n-3} + \cdots$

then, $f'''(a) = 2\cdot 3c_3 = 3!c_3$

so, $c_3 = \frac{f'''(a)}{3!}$

Do you see a pattern?

nth derivative of f(x)

Predict $f^{(n)}(a) = n!c_n$

so, $c_n = \frac{f^{(n)}(a)}{n!}$

Therefore, the Power Series:

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$

can now be written as the

Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

kth derivative

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

where f is generated (centered) @ $x = a$

and the partial sum, called the

notice: not to ∞

Taylor Polynomial of order n for $f(x)$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

notice: only to n, not to ∞

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

where f is generated (centered) @ $x = a$

For a more specific case, we have:

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k$$

where f is generated (centered) @ $x = 0$

Examples:

Find the Maclaurin Series of $f(x) = e^x$

$f(0) = e^0 = 1$ $\quad f'(x) = e^x$ $\quad f''(x) = e^x$

$f'(0) = e^0 = 1$ $\quad f''(0) = e^0 = 1$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \frac{x^n}{n!} + \cdots$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Find the Taylor Series for $f(x) = e^x$ at $x = 2$.

$f(2) = e^2$ $\quad f'(x) = e^x$ $\quad f''(x) = e^x$

$f'(2) = e^2$ $\quad f''(2) = e^2$

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \dots$$

$$= e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \dots + \frac{e^2}{n!}(x-2)^n + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}}$$

Construct a fifth order Taylor Polynomial for $\sin x$ at $x = 0$.

← 5th power/degree

$f(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$

$f(0) = \sin 0 = 0$ $\quad f'(0) = \cos 0 = 1$ $\quad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$ $\quad f^{(4)}(x) = \sin x$ $\quad f^{(5)}(x) = \cos x$

$f'''(0) = -\cos 0 = -1$ $\quad f^{(4)}(0) = \sin 0 = 0$ $\quad f^{(5)}(0) = \cos 0 = 1$

$$P_5(x) = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}}$$

Find a Taylor Polynomial of order 3 generated by $f(x) = \frac{1}{x}$ at $x = 1$.

← 3rd power/degree

$f(1) = 1$ $\quad f'(x) = -\frac{1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$ $\quad f'''(x) = \frac{-6}{x^4}$

$f'(1) = -1$ $\quad f''(1) = 2$ $\quad f'''(1) = -6$

$$P_3(x) = 1 + -1(x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3$$

$$\boxed{P_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3}$$