HW5 – AP Taylor Series Practice

1. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

a) Write a third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$\boxed{T_3(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 + -\frac{8}{6}(x-2)^3}$$

$$f(1.5) \approx T_3(1.5) = -3+5(1.5-2)+\frac{3}{2}(1.5-2)^2 - \frac{4}{3}(1.5-2)^3$$

$$\boxed{f(1.5) = -4.958}$$

replace in for x in $T_3(x)$

b) Write the fourth-degree Taylor polynomial, $P(x)$. For $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

$$P(x) = T_4(x) = -3 + 5(x^2+2-2) + \frac{3}{2}(x^2+2-2)^2$$

$$\boxed{P(x) = -3 + 5x^2 + \frac{3}{2}x^4}$$

← use $T_3(x)$ from part (a)... stop at 3rd term b/c has degree 4

g has rel. min when $g'(x)=0$ and $g''(x)>0$

$$P(x) = -3 + 5x^2 + \frac{3}{2}x^4$$

$$= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \ldots$$

no linear term, so $g'(0)=0$

$$\frac{g''(0)}{2!} = 5$$

so, $g''(0) = 10$

$\boxed{\therefore,\ \text{since } g'(0)=0 \text{ and } g''(0)>0,\ g \text{ has rel. min @ } x=0}$

2. The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$ and $f(5) = \frac{1}{2}$. Write the third-degree Taylor polynomial for f about $x = 5$.

$f(5) = \frac{1}{2}$

$f'(5) = \frac{(-1)^1 1!}{2^1(1+2)} = -\frac{1}{6}$

$f''(5) = \frac{(-1)^2 \cdot 2!}{2^2(2+2)} = \frac{2!}{4(4)}$

$f'''(5) = \frac{(-1)^3 \cdot 3!}{2^3(3+2)} = -\frac{3!}{8(5)}$

$$T_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{\frac{2!}{16}}{2!}(x-5)^2 + \frac{-\frac{3!}{8(5)}}{3!}(x-5)^3$$

$$\boxed{T_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3}$$

3. The Maclaurin series for the function f is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

a) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

watch out chain rule ☺

$$f'(x) = 2 + \frac{4 \cdot 2x}{2} + \frac{8 \cdot 3x^2}{3} + \frac{16 \cdot 4x^3}{4} + \cdots + \frac{(n+1)\cdot(2x)^n \cdot 2}{n+1} + \cdots$$

$$\boxed{f'(x) = 2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots}$$

b) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$

$$f'(-1/3) = 2 + 4(-1/3) + 8(-1/3)^2 + \cdots + 2(2\cdot -1/3)^n + \cdots$$
$$= 2 + 2(-2/3) + 2(-2/3)^2 + \cdots + 2(-2/3)^n + \cdots$$

geometric series, w/ $a=2$ and $r=-2/3$

so, $f'(-1/3) = \frac{2}{1 - -2/3}$

$= \frac{2}{5/3}$

$\boxed{f'(-1/3) = 6/5}$

OR

$f'(x) = \frac{2}{1-2x}$ where $a=2$, $r=2x$

so, $f'(-1/3) = \frac{2}{1-2(-1/3)}$

$= \frac{2}{5/3}$

$\boxed{f'(-1/3) = 6/5}$

4. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

a) Find $P(x)$.

$f(0) = \sin \pi/4 = \frac{\sqrt{2}}{2}$

$f'(x) = 5\cos(5x + \pi/4)$ $\quad f''(x) = -25\sin(5x + \frac{\pi}{4})$ $\quad f'''(x) = -125\cos(5x + \frac{\pi}{4})$

$f'(0) = \frac{5\sqrt{2}}{2}$ $\quad f''(0) = \frac{-25\sqrt{2}}{2}$ $\quad f'''(0) = -\frac{125\sqrt{2}}{2}$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$\boxed{P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3}$$

b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

$$P(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(5)x + \frac{-5^2\sqrt{2}}{2!\cdot 2}x^2 + \frac{-5^3\sqrt{2}}{3!\cdot 2}x^3 + \cdots + \frac{-5^{22}\sqrt{2}}{22!\cdot 2}x^{22} + \cdots$$

NOTICE: 1st two terms positive, next 2 negative

$$\boxed{\text{coefficient of } x^{22} = \frac{-5^{22}\sqrt{2}}{2(22!)}x^{22}}$$

c) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

$$G(x) = \int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{2\cdot 2!}t^2\right) dt$$

$$= \left(\frac{\sqrt{2}}{2}t + \frac{5\sqrt{2}}{2}\cdot\frac{1}{2}t^2 - \frac{25\sqrt{2}}{2\cdot 2!}\cdot\frac{1}{3}t^3\right)\Big|_0^x$$

$$\boxed{G(x) = \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3}$$