

9.3 Taylor's Theorem

Recall the Taylor Series, centered at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Recall Taylor's Polynomial approximation of $f(x)$:

$$\boxed{P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

then,

$$\begin{aligned} f(x) - P_n(x) &= \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \dots \\ &\leq \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \end{aligned}$$

if terms decreasing

Error/Remainder

$$R_n(x) = f(x) - P_n(x)$$

LaGrange Error Bound

$$\begin{aligned} |R_n(x)| &= |f(x) - P(x)| \\ &\leq \left| \frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1} \right| \\ &\leq \frac{\max |f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1} \quad \text{where } a < c < x \end{aligned}$$

$$\text{Lagrange Error Bound} \quad \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{where } |f^{(n+1)}(c)| \leq M, a < c < x$$

Example 1:

- Estimate the error of $\sin(0.2)$ from the Taylor Polynomial of order 4.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad \leftarrow \text{centered at } x=0$$

$$P_4(x) = x - \frac{x^3}{3!}$$

$$|R_4(x)| \leq \frac{\max |f^{(5)}(c)|}{5!} |x|^5 \quad \text{where } 0 < c < 0.2$$

$$\begin{aligned} |R_4(0.2)| &\leq \frac{1}{5!} (0.2)^5 \\ &\leq 2.667 \times 10^{-6} \end{aligned}$$

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ f^{(5)}(x) &= \cos x \end{aligned}$$

max $|f^{(5)}(x)|$ on $(0, 0.2)$
is 1 (magnitude of $\cos 1$)

Example 2:

Find the error bound for the 5th degree polynomial approximation of e^x .

$$|R_5(x)| \leq \frac{(\max |f^{(6)}(c)|)}{6!} |x|^6$$

$$|R_5(1)| \leq \frac{e}{6!} (1)^6$$

$$\leq \frac{e}{6!} \quad \leftarrow \text{non-calc. answer}$$

$$|R_5(1)| \leq 0.00377 \quad \leftarrow \text{calc. answer}$$

$$f(x) = e^x \quad @ x=1$$

$f'(x) = e^x$ (centered at $x=0$)

$$f''(x) = e^x$$

$$f'''(x) = e^x$$

$$f^{(4)}(x) = e^x$$

$\max |f^{(6)}(c)|$ on $0 < c < 1$
is e



Example 3:

The approximation $\ln(1+x) \approx x - \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation

Theorem to get a bound for the maximum error when $|x| \leq 0.1$.

$$|R_2(x)| \leq \frac{(\max |f^{(3)}(c)|)}{3!} |x|^3$$

$$\leq \frac{\frac{2}{3}}{3!} |-.1|^3$$

$$\leq \frac{2}{(0.9)^3 (0.1)^3 3!}$$

$$\leq .00457$$

max error is .00457

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

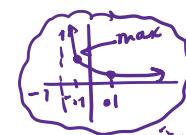
$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

Since $|x| \leq 0.1 \rightarrow [-0.1, 0.1]$

$\max |f^{(3)}(x)|$ is

$$\frac{2}{(1+(-0.1))^3} = \frac{2}{(0.9)^3}$$



Example 4:

What is the smallest order of Taylor polynomial centered at $x=1$ which will approximate e^{x-1} on the domain $-1 \leq x \leq 3$ with LaGrange error bound less than 1?

$$|R_n(x)| \leq \frac{\max |f^{(n+1)}(c)|}{(n+1)!} (x-1)^{n+1}$$

$$|R_n(3)| \leq \frac{e^2}{(n+1)!} |3-1|^{n+1}$$

$$\leq \frac{e^2}{(n+1)!} 2^{n+1} < 1$$

$$f'(x) = e^{x-1}$$

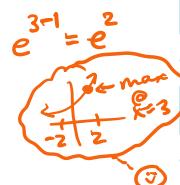
$$f''(x) = e^{x-1}$$

$$f'''(x) = e^{x-1}$$

since $-1 \leq x \leq 3$,

$$\max |f^{(n+1)}(x)| \approx e^{3-1} = e^2$$

$f(x)$ centered at $x=1$



test #s
for n
until < 1

$$\text{for } n=1, \frac{e^2}{2!} 2^2 > 1$$

$$n=2, \frac{e^2}{3!} 2^3 > 1$$

⋮

$$n=5, \frac{e^2}{6!} 2^6 < 1 \quad \checkmark$$