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Lagrange Error Bound Practice

1. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

$T_3(x)$ about $x = 2$ $= f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3$

$= -3 + 5(x-2) + \frac{3}{2}(x-2)^2 + \frac{-4}{3}(x-2)^3$ $\quad \frac{f'''(2)}{6} = \frac{-8}{6} = \frac{-4}{3}$

$f(1.5) \approx -3 + 5(1.5-2) + \frac{3}{2}(1.5-2)^2 - \frac{4}{3}(1.5-2)^3$

≈ -4.958

b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5,2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

$|R_3(x)| \le \frac{\max|f^{(4)}(x)|}{4!}|x-2|^4$

$|R_3(1.5)| \le \frac{3}{4!}|1.5-2|^4 = 0.0078125$

$f(1.5) - R < f(1.5) < f(1.5) + R$

$-4.958 - .0078125 < f(1.5) < -4.958 + .0078125$

$-4.966 < f(1.5) < -4.951$

$\therefore,\ f(1.5) \neq -5$

2. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$

$|f(x) - P_3(x)| \le \frac{\max|f^{(4)}(x)|}{4!}|x|^4$

$|f(\frac{1}{10}) - P(\frac{1}{10})| \le \frac{625}{4!}(\frac{1}{10})^4$

$\le \frac{1}{384}$

Since $\frac{1}{384} < \frac{1}{100}$,

$|f(\frac{1}{10}) - P(\frac{1}{10})| < \frac{1}{100}$

$f'(x) = 5\cos(5x + \frac{\pi}{4})$

$f''(x) = -25\sin(5x + \frac{\pi}{4})$

$f'''(x) = -125\cos(5x + \frac{\pi}{4})$

$f^{(4)}(x) = 625\sin(5x + \frac{\pi}{4})$

max of $f^{(4)}(x)$ is 625 b/c $|\sin(f(x))| \le 1$, so $f^{(4)}(x)$ is at most 625

3. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

a) Use $T(x)$ to find an approximation for $f(0)$.

$$T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$$

$$f(0) \approx T(0) = 7 - 9(0-2)^2 - 3(0-2)^3$$

$$\boxed{f(0) \approx -5}$$

b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 6$ for all x in the closed interval $[0,2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (a) to explain why $f(0)$ is negative.

$$|R_4(x)| \le \frac{\max|f^{(4)}(x)|}{4!}|x-2|^4$$

$$|R_4(0)| \le \frac{6}{4!}|0-2|^4$$

$$\le 4$$

$$f(0) - R \le f(0) \le f(0) + R$$

$$f(0) - 4 \le f(0) \le f(0) + 4$$

$$-5 - 4 \le f(0) \le -5 + 4$$

$$-9 \le f(0) \le -1 \quad , \quad \therefore f(0) \text{ is negative}$$