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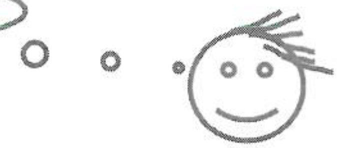
9.4 Radius of Convergence

Recall that a geometric series converges when $|r| < 1$



<u>Radius of Convergence Theorem</u>

Convergence for a Power Series, $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$,

occurs in one of 3 ways:

where R is the radius of convergence

① the series converges $\forall$x $(R = \infty)$

② $\exists$ a # R > 0 s.t. the series converges absolutely if $|x-a| < R$

and the series diverges absolutely if $|x-a| > R$

③ the series converges only at x = a $(R = 0)$

Find the radius of convergence and the interval of convergence.

Example 1

$$\sum_{n=0}^{\infty} 2^n x^{n+2} = \sum_{n=0}^{\infty} 2^n x^n x^2$$

$$= x^2 \sum_{n=0}^{\infty} (2x)^n$$

geometric, so $|r| < 1$

$|2x| < 1$

$|x| < \frac{1}{2}$

$|x - 0| < \frac{1}{2}$

radius of convergence is $\frac{1}{2}$
interval of convergence is: $-\frac{1}{2} < x < \frac{1}{2}$

Example 2

$$\sum_{n=0}^{\infty} (x+5)^n$$

geometric, so $|r| < 1$

$|x+5| < 1$

radius of convergence is 1

$-1 < x+5 < 1$

interval of convergence: $-6 < x < -4$

Example 3

$$\sum_{n=0}^{\infty} \frac{2^n}{n+1}$$

not geometric

but recall a series diverges if $\lim_{n\to\infty} a_n \neq 0$

$\lim_{n\to\infty} \frac{2^n}{n+1} = \frac{\infty}{\infty}$ ∴ L'Hôpital

$\lim_{n\to\infty} \frac{2^n \cdot \ln 2}{1} = \infty \neq 0$, so $\sum_{n=0}^{\infty} \frac{2^n}{n+1}$ diverges

Convergence Tests

Recall comparison test from section 8.4, if $0 \le f(x) \le g(x)$

1) and $\int g(x)dx$ converges, then $\int f(x)dx$ also converges.
2) and $\int f(x)dx$ diverges, then $\int g(x)dx$ also diverges.

Direct Comparison Test

If $\sum a_n$ has no negative terms,

① $a_n \le c_n$ and $\sum a_n$ converges, then $\sum c_n$ converges

② $d_n \le a_n$ and $\sum d_n$ diverges, then $\sum a_n$ diverges

Ratio Test

If $\sum a_n$ has no negative terms and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

then,

① if L < 1, then the series converges

② if L > 1, then the series diverges

③ if L = 1, then the test fails. ∴ (inconclusive test)

Determine if the series converges or diverges.

Example 1

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Direct Comparison Test

$n^2+1 > n^2$

$\frac{1}{n^2} > \frac{1}{n^2+1}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ also converges}}$

Example 2

$$\sum_{n=0}^{\infty} n^2 e^{-n}$$

Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot e^{-n} \cdot e^{-1}}{n^2 \cdot e^{-n}}$$

$$= \lim_{n\to\infty} \frac{1}{e} \frac{(n+1)^2}{n^2}$$

$$= \frac{1}{e}$$

$\frac{1}{e} < 1$, so $\boxed{\sum_{n=0}^{\infty} n^2 e^{-n} \text{ converges}}$

Example 3

$$\sum_{n=0}^{\infty} n!\,e^{-n}$$

* Ratio Test *

$$\lim_{n\to\infty} \frac{(n+1)!\,e^{-(n+1)}}{n!\,e^{-n}} = \lim_{n\to\infty} \frac{(n+1)n!\,e^{-n}\cdot e^{-1}}{n!\,e^{-n}}$$

$$= \lim_{n\to\infty} \frac{1}{e}\cdot(n+1)$$

$$= \infty$$

$\infty > 1$, so $\boxed{\sum_{n=0}^{\infty} n!\,e^{-n} \text{ diverges}}$

Find the radius and interval of convergence.

Example 1

$$\sum_{n=0}^{\infty} \frac{n}{2^n}(x-3)^n$$

* Ratio Test *

$$\lim_{n\to\infty} \frac{\frac{(n+1)}{2^{n+1}}(x-3)^{n+1}}{\frac{n}{2^n}(x-3)^n} = \lim_{n\to\infty} \frac{(n+1)(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x-3)^n}$$

$$= \lim_{n\to\infty} \frac{x-3}{2}\cdot\frac{(n+1)}{n}$$

$$= \frac{x-3}{2}$$

converges if $\left|\frac{x-3}{2}\right| < 1$

$|x-3| < 2$

$\boxed{\text{radius of convergence is: } 2}$

$-2 < x-3 < 2$

$\boxed{1 < x < 5 \quad \text{interval of convergence}}$ ← test endpts

Test Endpts

@ $x=1$ $\sum_{n=0}^{\infty} \frac{n}{2^n}(-2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot n$

$\lim_{n\to\infty} (-1)^n n$ diverges

@ $x=5$, $\sum_{n=0}^{\infty} \frac{n}{2^n}(2)^n = \sum_{n=0}^{\infty} n$

$\lim_{n\to\infty} n$ diverges

∴, endpts not included

Example 2

$$\sum_{n=1}^{\infty} n!\,(x-2)^n$$

* Ratio Test *

$$\lim_{n\to\infty} \frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n} = \lim_{n\to\infty} (n+1)(x-2)$$

$= \infty$ {when $x \neq 2$} ...

when $x = 2$ $\lim_{n\to\infty} (n+1)(x-2) = 0$

$\boxed{\text{radius of convergence } 0}$

$\boxed{\text{interval of convergence : } [2]}$