9.4 Sequences & Series

Target 7D: Calculate the sums of finite and infinite series

Review of Prior Concepts

1. Find the value of (without calculator):

$$\sum_{k=5}^{9}(11-3k) = 11-3(5)+11-3(6)+11-3(7)+11-3(8)+11-3(9)$$
$$= -4-7-10-13-16 = \boxed{-50}$$

2. Find the value of (with calculator):

$$\sum_{k=0}^{12}\left(\frac{1}{2}\right)^k = \boxed{\frac{8191}{4096}}$$

3. Find the sum of: $2+1+\frac{1}{2}+\frac{1}{4}+\cdots$

sum one term → $S_1 = 2$
sum 2 terms → $S_2 = 2+1 = 3$
$S_3 = 2+1+\frac{1}{2} = 3.5$
$S_4 = 2+1+\frac{1}{2}+\frac{1}{4} = 3.75$
$S_5 = 2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 3.875$
$S_6 = 3.9375$
$S_7 = 3.96875$
$\vdots$
$\approx \boxed{4}$

4. Find the sum of: $1+2+4+8+16+\cdots$

$S_1 = 1$
$S_2 = 1+2 = 3$
$S_3 = 1+2+4 = 7$
$S_4 = 1+2+4+8 = 15$
$S_5 = 31$
$S_6 = 63$
$\vdots$
$\approx \boxed{\infty}$

More Practice

Arithmetic & Geometric Sequences and Series

https://www.mathsisfun.com/algebra/sequences-sums-arithmetic.html
http://www.mathsisfun.com/algebra/sequences-sums-geometric.html
http://www.purplemath.com/modules/series4.htm

SAT Connection

Passport to Advanced Math

10. Interpret parts of nonlinear expressions in terms of their context. .

Example:

Of the following four types of savings account plans, which option would yield exponential growth of the money in the account?

A) Each successive year, 2% of the initial savings is added to the value of the account.

B) Each successive year, 1.5% of the initial savings and $100 is added to the value of the account.

C) Each successive year, 1% of the current value is added to the value of the account.

D) Each successive year, $100 is added to the value of the account.

exponential growth → explicit $a_n = a_1(r)^{n-1}$ or recursive $a_n = a_{n-1}\cdot r$

(A) $a_n = 0.2(a_1) + a_{n-1}$

(B) $a_n = 100 + 0.15(a_1) + a_{n-1}$

(C) $a_n = 0.1(a_{n-1}) + a_{n-1}$
$= a_{n-1}(0.1+1)$
$= 1.1\,a_{n-1}$ → matches geometric recursive rule

(D) $a_n = 100 + a_{n-1}$

Solution

Finite Geometric Series

$$\sum_{k=1}^{n} a_1 r^{k-1} = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n$$

Formula for Sum of the Terms in a Finite Geometric Sequence is:

$$\boxed{\sum_{k=1}^{n} a_1 r^{k-1} = \frac{a_1(1-r^n)}{1-r}}$$

Example 1: Find the sum of the sequence: $\overset{a_1}{2}, 6, 18, \ldots, \overset{a_n}{2\cdot 3^8}$

geometric w/ $r = 3$

$\frac{6}{2} = 3$, $\frac{18}{6} = 3$ … ☺

$a_n = a_1 r^{n-1}$

$2\cdot 3^8 = 2(3)^{n-1}$ ← divide by 2

$3^8 = 3^{n-1}$ ← bases equal, so exponents equal

$8 = n-1$

$9 = n$

$$\sum_{k=1}^{9} 2\cdot 3^{n-1} = \frac{2(1-3^9)}{1-3} = \frac{2(1-3^9)}{-2} = \boxed{19682}$$

Example 2: Find the sum: $\overset{a_1}{4} - \frac{4}{3} + \frac{4}{9} - \frac{4}{27} + \overset{a_n}{\frac{4}{81}}$

count to 5 terms

geometric w/ $r = -\frac{1}{3}$

$\frac{-4/3}{4} = -\frac{1}{3}$, $\frac{4/9}{-4/3} = -\frac{1}{3}$ … ☺

$$\sum_{k=1}^{5} 4\left(-\tfrac{1}{3}\right)^{k-1} = \frac{4(1-(-1/3)^5)}{1 - -1/3} = \frac{4\left(1-\left(-\frac{1}{243}\right)\right)}{1+1/3} = \frac{4\left(1+\frac{1}{243}\right)}{4/3} = \boxed{\frac{244}{81}} \text{ or } \boxed{3.01235}$$

Infinite Geometric Series

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n + \cdots$$

Formula for Sum of the Terms of an Infinite Geometric Sequence is:

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r} \qquad \text{when } |r| < 1}$$

Example 1: Find the sum of the sequence: $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$

geometric w/ $r = \frac{1}{2}$

$\frac{1}{2} = \frac{1}{2}$, $\frac{1/2}{1} = 1/2$ … ☺

Is $|\frac{1}{2}| < 1$?

$\frac{1}{2} < 1$ ✓

$$\sum_{k=1}^{\infty} 2\left(\tfrac{1}{2}\right)^{k-1} = \frac{2}{1-1/2} = \frac{2}{1/2} = 2\cdot\frac{2}{1} = \boxed{4}$$

The series "converges to 4"

Example 2: Find the sum: $\frac{1}{48}+\frac{1}{16}+\frac{3}{16}+\frac{9}{16}+\cdots$

geometric w/ $r=3$

Is $|3|<1$?

$3<1$ ✗

$\frac{\frac{1}{16}}{\frac{1}{48}}=\frac{1}{16}\cdot\frac{48}{1}=3$

$\frac{3/16}{1/16}=\frac{3}{16}\cdot\frac{16}{1}=3$

Since $|3| \nless 1$, the sum is ∞.

"The series diverges"

Now, you try…

1. Find the sum of the sequence: $3,\ \frac{3}{4},\ \frac{3}{16},\ \frac{3}{64}, \ldots$

geometric w/ $r=\frac{1}{4}$

Is $|\frac{1}{4}|<1$?

$\frac{1}{4}<1$ ✓

$$\sum_{k=1}^{\infty} 3\left(\tfrac{1}{4}\right)^{k-1} = \frac{3}{1-1/4} = \frac{3}{3/4} = 3\cdot\frac{4}{3} = \boxed{4}$$

The series converges to 4

2. Find:

$$\sum_{k=1}^{\infty}(-2)^k = (-2)^0+(-2)^1+(-2)^2+\cdots = 1-2+4+\cdots$$

geometric w/ $r=-2$

Is $|-2|<1$?

$2<1$ ✗

$\frac{-2}{1}=-2$

$\frac{4}{-2}=-2$

Since $|-2| \nless 1$, the series diverges.

3. Find the sum: $\frac{1}{16}-\frac{1}{48}+\frac{1}{144}-\cdots$

geometric w/ $r=\frac{-1}{3}$

Is $|-\frac{1}{3}|<1$?

$\frac{1}{3}<1$ ✓

$\frac{-1/48}{1/16}=\frac{-1}{48}\cdot\frac{16}{1}=\frac{-1}{3}$

$\frac{1/144}{-1/48}=\frac{1}{144}\cdot\frac{-48}{1}=\frac{-1}{3}$

$$\sum_{k=1}^{\infty}\frac{1}{16}\left(\tfrac{1}{3}\right)^{k-1} = \frac{1/16}{1-^-1/3} = \frac{1/16}{4/3} = \frac{1}{16}\cdot\frac{3}{4} = \boxed{\frac{3}{64}}$$

The series converges to $\frac{3}{64}$

More Practice

Geometric Series
http://www.purplemath.com/modules/series5.htm
https://www.khanacademy.org/math/algebra2/sequences-and-series/copy-of-geometric-sequence-series/v/geometric-series-introduction
http://www.mathsisfun.com/algebra/sequences-sums-geometric.html
https://youtu.be/yYxzq_O18Mg
https://youtu.be/-JH5XSvJFTA
https://youtu.be/DO1bIuqFIDQ
https://youtu.be/haK3oC0L_a8

Homework Assignment

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SAT Connection

Solution

Choice C is correct. Let I be the initial savings. If each successive year, 1% of the current value is added to the value of the account, then after 1 year, the amount in the account will be $I + 0.01I = I(1 + 0.01)$; after 2 years, the amount in the account will be $I(1+0.01)+0.01I(1+0.01)=(1+0.01)I(1+0.01)=I(1+0.01)^2$; and after t years, the amount in the account will be $I(1+0.01)^t$. This is exponential growth of the money in the account.

Choice A is incorrect. If each successive year, 2% of the initial savings, I, is added to the value of the account, then after t years, the amount in the account will be $I + 0.02It$, which is linear growth. Choice B is incorrect. If each successive year, 1.5% of the initial savings, I, and \$100 is added to the value of the the account, then after t years the amount in the account will be $I + (0.015I + 100)t$, which is linear growth. Choice D is incorrect. If each successive year, \$100 is added to the value of the account, then after t years the amount in the account will be $I + 100t$, which is linear growth.