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9.5 Testing Convergence at Endpoints

More Convergence Tests

Integral Test

Suppose a_n is a sequence of positive terms and $a_n = f(n)$, where f is continuous, positive, decreasing function of $x \ \forall \ x \geq N$ (where N is positive integer),

for all

① If $\int_N^\infty f(x)\,dx$ converges, then $\sum_{n=N}^\infty a_n$ converges

② If $\int_N^\infty f(x)\,dx$ diverges, then $\sum_{n=N}^\infty a_n$ diverges

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Determine if the series converges or diverges.

Example 1

$$\sum_{n=1}^\infty \frac{1}{1+10n}$$

$a_n = \frac{1}{1+10n}$

$N = 1$

Integral Test

$$\int_1^\infty \frac{1}{1+10x}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{1+10x}\,dx = \lim_{a\to\infty}\left(\frac{1}{10}\ln|1+10x|\right)\Big|_1^a$$

$$= \lim_{a\to\infty}\left(\frac{1}{10}\ln|1+10a| - \frac{1}{10}\ln|11|\right)$$

$$= \infty \text{ diverges}$$

Since $\int_1^\infty \frac{1}{1+10x}\,dx$ diverges, then $\boxed{\sum_{n=1}^\infty \frac{1}{1+10n} \text{ also diverges}}$

Example 2

$$\sum_{n=1}^\infty \frac{1}{n^3}$$

$a_n = \frac{1}{n^3}$

$N = 1$

Integral Test

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{a\to\infty}\int_1^a x^{-3}\,dx = \lim_{a\to\infty}\left(-\frac{1}{2}x^{-2}\right)\Big|_1^a$$

$$= \lim_{a\to\infty}\left(-\frac{1}{2a^2} - -\frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2} \text{ converges}$$

Since $\int_1^\infty \frac{1}{x^3}\,dx$ converges, then $\boxed{\sum_{n=1}^\infty \frac{1}{n^3} \text{ also converges}}$

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p-Series Test

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

① if $p > 1$, then the series converges

② if $0 < p < 1$, then the series diverges

③ if $p = 1$, then the series diverges

Determine if the series converges or diverges.

Example 1 *p-series test*

$$\sum_{n=1}^{\infty} n^{-2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ since $p>1$, then $\boxed{\sum_{n=1}^{\infty} n^{-2} \text{ converges}}$

Example 2 * p-series test *

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Since $p=1$, then $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ $\forall\, n \geq N$ (where N is positive integer)

a series another series

① If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ (where $0 < c < \infty$) and $\sum b_n$ converges, then $\sum a_n$ converges

or if $\sum b_n$ diverges, then $\sum a_n$ diverges

RECALL: When does $\lim_{n\to\infty} \frac{f(x)}{g(x)} = \#$?
this means that exp N = exp D
so look @ coefficients ... ☺

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

means that
$b_n > a_n$
when Larger series converges,
then the smaller series also converges ... ☺

③ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Means $b_n < a_n$ when smaller series diverges, then larger series also diverges ... ☺

Determine if the series converges or diverges.

Example 1 *Limit Comparison Test*

$$\sum_{n=1}^{\infty} \frac{3}{n+1}$$

$a_n = \frac{3}{n+1}$ compares to $b_n = \frac{3}{n}$

$\lim_{n\to\infty} \frac{\frac{3}{n+1}}{\frac{3}{n}} = \lim_{n\to\infty} \frac{3}{n+1} \cdot \frac{n}{3} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ ← constant, so

Since $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges (by p-series test $p=1$), then $\boxed{\sum_{n=1}^{\infty} \frac{3}{n+1} \text{ also diverges}}$

Example 2 *Limit Comparison Test*

$$\sum_{n=1}^{\infty} \frac{5n^3 - 3n}{n^2(n+2)(n^2+5)}$$

$a_n = \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ compares to $b_n = \frac{n^3}{n^5}$ or $\frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{5n^3-3n}{n^2(n+2)(n^2+5)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{5n^3-3n}{(n+2)(n^2+5)} = 5$ converges

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by p-series test $p>1$), then $\boxed{\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)} \text{ also converges}}$