AP Series Problems

1. A function \( f \) is defined by \( f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^n+1}x^n + \cdots \) for all \( x \) in the interval of convergence of the given power series. Find the interval of convergence for this power series. Show the work that leads to your answer.

2. The function \( f \) has a Taylor series about \( x = 2 \) that converges to \( f(x) \) for all \( x \) in the interval of convergence. The \( n \)th derivative of \( f \) at \( x = 2 \) is given by \( f^{(n)}(x) = \frac{(n+1)!}{3^n} \) for \( n \geq 1 \), and \( f(2) = 1 \).
   a) Write the first four terms and the general term of the Taylor series for \( f \) about \( x = 2 \).
   b) Find the radius of convergence for the Taylor series for \( f \) about \( x = 2 \). Show the work that leads to your answer.
   c) Let \( g \) be a function satisfying \( g(2) = 3 \) and \( g'(x) = f(x) \) for all \( x \). Write the first four terms and the general term of the Taylor series for \( g \) about \( x = 2 \).
   d) Does the Taylor series for \( g \) as defined in part (c) converge at \( x = -2 \)? Give a reason for your answer.

3. Let \( f \) be a function with derivatives of all orders and for which \( f(2) = 7 \). When \( n \) is odd, the \( n \)th derivative of \( f \) at \( x = 2 \) is 0. When \( n \) is even and \( n \geq 2 \), the \( n \)th derivative of \( f \) at \( x = 2 \) is given by \( f^{(n)}(2) = \frac{(n-1)!}{3^n} \).
   a) Write the sixth-degree Taylor polynomial for \( f \) about \( x = 2 \).
   b) In the Taylor series for \( f \) about \( x = 2 \), what is the coefficient of \( (x - 2)^2n \) for \( n \geq 1 \) ?
   c) Find the interval of convergence of the Taylor series for \( f \) about \( x = 2 \). Show the work that leads to your answer.

4. The function \( f \) is defined by the power series
   \[
   f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n + 1} + \cdots
   \]
   for all real numbers \( x \) for which the series converges. The function \( g \) is defined by the power series
   \[
   g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots
   \]
   for all real numbers \( x \) for which the series converges.
   Find the interval of convergence of the power series for \( f \). Justify your answer.

5. The Maclaurin series for \( \ln \left( \frac{1}{1-x} \right) \) is \( \sum_{n=1}^{\infty} \frac{x^n}{n} \) with interval of convergence \( -1 \leq x < 1 \).
   a) Find the Maclaurin series for \( \ln \left( \frac{1}{1+3x} \right) \) and determine the interval of convergence.
   b) Find the value of \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)
   c) Give a value of \( p \) such that \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges, but \( \sum_{n=1}^{\infty} \frac{1}{n^2p} \) converges. Give reasons why your value of \( p \) is correct.