AP Series Problems (Solutions)

1.

$$\lim_{n\to\infty}\left|\frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)}{(n+1)}\frac{x}{3}\right| = \left|\frac{x}{3}\right| < 1$$

At $x=-3$, the series is $\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3}$, which diverges.

At $x=3$, the series is $\sum_{n=0}^{\infty}\frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

4 :
- 1 : sets up ratio test
- 1 : computes limit
- 1 : conclusion of ratio test
- 1 : endpoint conclusion

2.

(a) $f(2)=1;\ f'(2)=\frac{2!}{3};\ f''(2)=\frac{3!}{3^2};\ f'''(2)=\frac{4!}{3^3}$

$$f(x) = 1+\frac{2}{3}(x-2)+\frac{3!}{2!3^2}(x-2)^2+\frac{4!}{3!3^3}(x-2)^3+ \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n+\cdots$$

$$= 1+\frac{2}{3}(x-2)+\frac{3}{3^2}(x-2)^2+\frac{4}{3^3}(x-2)^3+ \cdots + \frac{n+1}{3^n}(x-2)^n+\cdots$$

3 :
- 1 : coefficients $\frac{f^{(n)}(2)}{n!}$ in first four terms
- 1 : powers of $(x-2)$ in first four terms
- 1 : general term

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \frac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 :
- 1 : sets up ratio
- 1 : limit
- 1 : applies ratio test to conclude radius of convergence is 3

(c) $g(2)=3;\ g'(2)=f(2);\ g''(2)=f'(2);\ g'''(2)=f''(2)$

$$g(x) = 3+(x-2)+\frac{1}{3}(x-2)^2+\frac{1}{3^2}(x-2)^3+ \cdots + \frac{1}{3^n}(x-2)^{n+1}+\cdots$$

2 :
- 1 : first four terms
- 1 : general term

(d) No, the Taylor series does not converge at $x=-2$ because the geometric series only converges on the interval $|x-2|<3$.

1 : answer with reason

3.

(a) $P_6(x) = 7 + \frac{1!}{3^2}\cdot\frac{1}{2!}(x-2)^2 + \frac{3!}{3^4}\cdot\frac{1}{4!}(x-2)^4 + \frac{5!}{3^6}\cdot\frac{1}{6!}(x-2)^6$

$3: \begin{cases} 1: \text{polynomial about } x = 2 \\ 2: P_6(x) \\ \quad \langle -1 \rangle \text{ each incorrect term} \\ \quad \langle -1 \rangle \text{ max for all extra terms,} \\ \qquad + \cdots, \text{ misuse of equality} \end{cases}$

(b) $\frac{(2n-1)!}{3^{2n}}\cdot\frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$f(x) = 7 + \sum_{n=1}^{\infty}\frac{1}{2n\cdot 3^{2n}}(x-2)^{2n}.$

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{2(n+1)}\cdot\frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n}\cdot\frac{1}{3^{2n}}(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot\frac{3^{2n}}{3^2 3^{2n}}(x-2)^2\right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty}\frac{3^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty}\frac{(-3)^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

$5: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{identifies interior of interval of convergence} \\ 1: \text{considers both endpoints} \\ 1: \text{analysis/conclusion for both endpoints} \end{cases}$

4.

(a) $\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)}\cdot|x|$

$\lim_{n\to\infty}\frac{(n+1)^2}{(n+2)(n)}\cdot|x| = |x|$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

$5: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{identifies radius of convergence} \\ 1: \text{considers both endpoints} \\ 1: \text{analysis/conclusion for both endpoints} \end{cases}$

5.

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n}$ or $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

2 { 1 : series
  1 : interval of convergence

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

3 { 1 : correct p
  1 : reason why $\sum \frac{1}{n^p}$ diverges
  1 : reason why $\sum \frac{1}{n^{2p}}$ converges