DATE: ________________

9.5 Testing Convergence at Endpoints (continued)

One More Convergence Test

Alternating Series Test

The series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if ALL three of the following conditions are true:

① each $u_n > 0$ — u_n's are positive ... ☺

② $u_n \geq u_{n+1}\ \forall\ n \geq N$ (where *N* is some integer)

if $u_n \geq u_{n+1}$, this means that the terms are getting smaller (series is decreasing → $f'(x) < 0\ \forall x$) ... ☺

③ $\lim_{n\to\infty} u_n = 0$ — even though alternating btn pos. + neg, the terms are going toward zero ... ☺

Determine if the series converges or diverges.

Example 1

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$$

Alternating Series Test

① $\frac{1}{\ln n} > 0$ for $2 < n < \infty$

so, $u_n > 0$ ✓

② $\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$?

since $\ln(n+1) > \ln(n)$ is true, then

$\frac{1}{\ln n} > \frac{1}{\ln(n+1)}$

so, $u_n > u_{n+1}$ ✓

③ $\lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{\infty} = 0$

so $\lim_{n\to\infty} u_n = 0$ ✓

∴, $\boxed{\sum_{n=2}^{\infty} (-1)^n \cdot \frac{1}{\ln n} \text{ converges}}$

Example 2

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

* Alternating Series Test *

① $\frac{1}{\sqrt{n}} > 0$ for $1 < n < \infty$

so $u_n > 0$ ✓

② $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$?

let $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$f'(x) = -\frac{1}{2}x^{-3/2} < 0$ for $1 < x < \infty$

so $f(x)$ dec

so, $u_n > u_{n+1}$ ✓

③ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0$

so $\lim_{n\to\infty} u_n = 0$ ✓

∴, $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges}}$

2 Types of Convergence

① Absolute convergence: If $\sum|a_n|$ converges, then $\sum a_n$ converges absolutely

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$a_n = \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test $(p>1)$,

then $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges absolutely}}$

② Conditional convergence: If $\sum|a_n|$ diverges but $\sum a_n$ converges,

then $\sum a_n$ converges conditionally

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$a_n = \frac{(-1)^{n+1}}{n}$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test $(p=1)$, so now check if $\sum a_n$ converges...

* Alternating Series Test *

① $\frac{1}{n} > 0$ for $1 < n < \infty$

so $u_n > 0$ ✓

② $n+1 > n$

$\frac{1}{n} > \frac{1}{n+1}$

so $u_n > u_{n+1}$ ✓

③ $\lim_{n\to\infty} \frac{1}{n} = 0$

so $\lim_{n\to\infty} u_n = 0$ ✓

so $\sum \frac{(-1)^{n+1}}{n}$ converges

$\therefore \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges conditionally}}$

Determine if the series converges absolutely, converges conditionally, or diverges.

Example 1

$$\sum_{n=1}^{\infty} (-1)^n n^2 \left(\frac{2}{3}\right)^n$$

$\sum_{n=1}^{\infty} \left|(-1)^n \cdot n^2 \cdot \left(\frac{2}{3}\right)^n\right| = \sum_{n=1}^{\infty} n^2 \left(\frac{2}{3}\right)^n$

* Ratio Test *

$\lim_{n\to\infty} \frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{2}{3} \frac{(n+1)^2}{n^2} = \frac{2}{3} < 1$, so converges.

$\therefore, \boxed{\sum (-1)^n n^2 \left(\frac{2}{3}\right)^n \text{ converges absolutely}}$

Example 2

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-test $(0<p<1)$, so now check if $\sum a_n$ converges...

* Alternating Series Test *

① $\frac{1}{\sqrt{n}} > 0$ for $1 < n < \infty$

so $u_n > 0$ ✓

② let $f(x) = x^{-1/2}$

$f'(x) = -\frac{1}{2}x^{-3/2} < 0$ for $1 < x < \infty$

($f(x)$ dec)

so $u_n > u_{n+1}$ ✓

③ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

so $\lim_{n\to\infty} u_n = 0$ ✓

so, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges

$\therefore \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges conditionally}}$