DATE: ___________

Convergence Practice

1. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-test $p>1$, converges

II. $\sum_{n=1}^{\infty} \frac{1}{n}$ p-test $p=1$, diverges

III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ *Alternating Series Test*

① $\frac{1}{\sqrt{n}} > 0$ for $1 < n < \infty$ so $u_n > 0$ ✓

② $\sqrt{n+1} > \sqrt{n}$

$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ so, $u_n > u_{n+1}$ ✓

③ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges

(A) I only

(B) III only

(C) I and II only

$\boxed{\textbf{(D)}}$ I and III only

(E) I, II, and III

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2. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges?

Ratio Test $\lim_{n\to\infty} \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} = \lim_{n\to\infty} (x-1)\left(\frac{n}{n+1}\right)$

$= x-1$

$|x-1|<1$

$-1<x-1<1$

$0 \le x < 2$

check endpts.

$x=0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

AST

① $\frac{1}{n} > 0$ ✓ ③ $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

② $n+1 > n$

$\frac{1}{n} > \frac{1}{n+1}$ ✓ so $x=0$ included

$x=2$ $\sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test

so, $x=2$ not included

(A) $-1 \le x < 1$

(B) $-1 \le x \le 1$

(C) $0 < x < 2$

$\boxed{\textbf{(D)}}$ $0 \le x < 2$

(E) $0 \le x \le 2$

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3. Which of the following series converge?

I. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$

II. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$

III. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

AST

✓ I. ① $\frac{1}{2n+1} > 0$ ✓

② $f(x) = \frac{1}{2x+1}$

$f'(x) = -(2x+1)^{-2} < 0$ ✓

③ $\lim_{n\to\infty} \frac{1}{2n+1} = 0$ ✓ so $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$ converge

✗ II. *Ratio Test*

$\lim_{n\to\infty} \frac{\frac{1}{n+1}\left(\frac{3}{2}\right)^{n+1}}{\frac{1}{n}\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)\left(\frac{n}{n+1}\right)$

$= 3/2 > 1$ so $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$ diverges

✗ III. *Integral Test*

$\int_2^{\infty} \frac{1}{x \ln x} dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$x\,du = dx$

$= \lim_{a\to\infty} \int_2^a \frac{1}{x \ln x} dx$

$= \lim_{a\to\infty} \int_{\ln 2}^{\ln a} \frac{1}{x} \frac{1}{u} \cdot x\,du$

$= \lim_{a\to\infty} \int_{\ln 2}^{\ln a} \frac{1}{u} du$

$= \lim_{a\to\infty} \left(\ln|u|\right)\Big|_{\ln 2}^{\ln a}$

$= \lim_{a\to\infty} \left(\ln|\ln a| - \ln|\ln 2|\right)$

$= \infty$

so $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

$\boxed{\textbf{(A)}}$ I only

(B) II only

(C) III only

(D) I and III only

(E) I, II, and III

2002BC FRQ

4. The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

b) Find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

c) Give a value of p such that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

d) Give a value of p such that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

a) $\ln\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$

$\ln\left(\frac{1}{1-(-3x)}\right) = -3x + \frac{(-3x)^2}{2} + \cdots + \frac{(-3x)^n}{n} + \cdots$

$\boxed{\ln\left(\frac{1}{1+3x}\right) = \sum_{n=1}^{\infty}\frac{(-3x)^n}{n}}$

$-1 \le -3x < 1$

$\frac{1}{3} \ge x > \frac{1}{-3}$

$\boxed{\text{interval of convergence: } -\frac{1}{3} < x \le \frac{1}{3}}$

b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right)$

$= \boxed{\ln\left(\frac{1}{2}\right)}$ or $\ln 1 - \ln 2$

$-\ln 2$

c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$

AST

① $\frac{1}{n^p} > 0 \quad \forall p$

② $f(x) = \frac{1}{x^p}$

$= -px^{-p-1} < 0$ when $-p < 0$

$p > 0$

③ $\lim_{n\to\infty}\frac{1}{n^p} = 0$ when $p > 0$

converges when $p > 0$

$\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ diverges when

$2p \le 1$

$p \le \frac{1}{2}$

so, $\boxed{0 < p \le \frac{1}{2}}$

d) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges when $0 < p \le 1$ (p-series test)

$\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ converges when $2p > 1$

$p > 1/2$

so, $\boxed{\frac{1}{2} < p \le 1}$