CollegeBoard
Advanced Placement Program

AP® Calculus AB
2009 Free-Response Questions
Form B

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CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

1. At a certain height, a tree trunk has a circular cross section. The radius $R(t)$ of that cross section grows at a rate modeled by the function

area of circle

$$\frac{dR}{dt} = \frac{1}{16}\left(3 + \sin\left(t^2\right)\right) \text{ centimeters per year}$$

for $0 \le t \le 3$, where time t is measured in years. At time $t = 0$, the radius is 6 centimeters. The area of the cross section at time t is denoted by $A(t)$.

→ initial radius (condition)

(a) Write an expression, involving an integral, for the radius $R(t)$ for $0 \le t \le 3$. Use your expression to find $R(3)$.

↳ $\int \frac{dR}{dt}$

$$R(t) = 6 + \int_0^t \frac{dR}{dt}\,dt$$

$$R(t) = 6 + \int_0^t \frac{1}{16}\left(3 + \sin(x^2)\right)dx$$

1pt - integral
1pt - $R(t)$

$$R(3) = 6 + \int_0^3 \frac{1}{16}\left(3 + \sin(x^2)\right)dx$$

$$R(3) = 6.611 \text{ cm}$$

1pt - $R(3)$

(b) Find the rate at which the cross-sectional area $A(t)$ is increasing at time $t = 3$ years. Indicate units of measure.

↳ $\frac{dA}{dt}$

↳ $A(t) = \pi r^2$

1pt - $A(t)$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

1pt - $A'(t)$

$$A'(t) = 2\pi R(t) \cdot \frac{dR}{dt}$$

$$A'(3) = 2\pi R(3) \cdot \left.\frac{dR}{dt}\right|_{t=3}$$

$$= 8.858 \text{ cm}^2/\text{yr}$$

1pt - answer w/ units

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Continue problem 1 on page 5.

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(c) Evaluate $\int_0^3 A'(t)\,dt$. Using appropriate units, interpret the meaning of that integral in terms of cross-sectional area.

$\int_0^3 A'(t)\,dt = A(t)\Big|_0^3$ 1pt - $A(t)\Big|_0^3$ (my 2nd FTC) ☺

$= A(3) - A(0)$

$= \pi(R(3))^2 - \pi(R(0))^2$

$= 24.201 \text{ cm}^2$ 1pt - answer

($A(t) = \pi(R(t))^2 = \pi r^2$ ⋮ ☺)

The area of the cross section increases by 24.201cm^2 from $t = 0$ to $t = 3$ years

OR The difference of the area of the cross section from $t = 0$ to $t = 3$ years is 24.201cm^2

1pt - interpretation

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2. A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

rate of distance

initial condition

(a) What was the distance between the road and the edge of the water at the end of the storm?

$\int$ rate of distance

distance btw road + edge of water @ $t = 5$

$= 35 + \int_0^5 f(t)\,dt$

1 pt - integral

1 pt - answer

$= 26.495$ meters

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(b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

$f'(4)$ = positive #

$\therefore$, $f(t)$ @ $t=4$ is inc

$f(t)$ → rate of distance

Rate of distance btw road + edge of water was changing @ $t = 4$ is increasing @ 1.007 meters/hr^2

1 pt - meaning of $f'(4)$

1 pt - units

☺

units:
$f(t)$ → m/hr
$f'(t)$ → $\frac{\text{m/hr}}{\text{hr}}$

Continue problem 2 on page 7.

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(c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

f(t) most negative → f(t) abs min

check crit #s & endpts in original → f(t)

$f'(t) = 0$ @ $t = 0.662$, $t = 2.840$

1 pt → $f'(t) = 0$

crit #s: $f(0.662) = -1.398$, $f(2.840) = -2.270$ → Distance btn road & edge of water dec most rapidly @ $t = 2.840$ hrs

endpts: $f(0) = -2$, $f(5) = -.486$

1 pt - answer

1 pt - reason

(d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

Sand pumped in + Sand washed away = 0 ← back to no difference in sand in vs. sand out

$$\int_0^5 f(t)\,dt + \int_0^x g(p)\,dp = 0$$

1 pt - integral of g

1 pt - answer

OR Original distance = distance at end of storm + Sand pumped in

$$35 = 35 + \int_0^5 f(t)\,dt + \int_0^x g(p)\,dp$$

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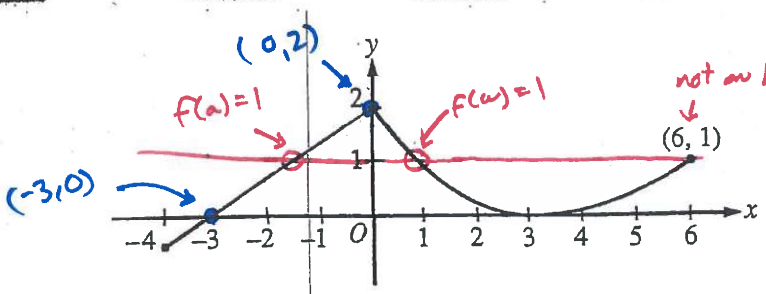

Graph of f

3. A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

↳ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ or $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

derivative on left

$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \frac{2-0}{0-3} = \frac{2}{3}$

1pt - difference quotient

derivative at right

$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} < 0$

$\therefore, \lim_{x\to 0^-} f'(x) \neq \lim_{x\to 0^+} f'(x),$

1pt - answer w/ reason

so, f is not diff'able @ x=0

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0? Give a reason for your answer.

$\frac{f(b)-f(a)}{b-a}$ a b

$\frac{f(6)-f(a)}{6-a} = 0$

1pt - avg rate of change

$f(6) - f(a) = 0$

$1 - f(a) = 0$

$1 = f(a)$

$f(a) = 1$ twice on $[-4, 6)$

1pt - answer w/ reason

$\therefore$ there are 2 values of a on $[-4, 6)$

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Continue problem 3 on page 9.

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(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

→ cont, diff'able

f is cont & diff'able on [a, 6) b/c f is twice-diff'able on (0, 6)

on [a, 6]

$f'(c) = \frac{1}{3}$

$\frac{F(6) - f(a)}{6 - a} = \frac{1}{3}$

$\frac{1 - f(a)}{6 - a} = \frac{1}{3}$

$\frac{1 - f(3)}{6 - 3}$

$\frac{1 - 0}{3} = \frac{1}{3}$

1 - f(a) = 1
6 - a = 3

If a = 3, 6 - a = 3 ✓
if a = 3, 1 - f(a) 1 - f(3) 1 - 0 = 1 ✓

1pt - yes and a = 3

1pt - reason

So, by MVT, ∃ a value c on (3, 6) s.t. $f'(c) = \frac{1}{3}$

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

g concave up → g'' > 0

$g'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$

$= f(x)$

$g''(x) = f'(x)$

$f'(x) > 0$ on $(-4, 0) \cup (3, 6)$ b/c f inc on $(-4, 0) \cup (3, 6)$

1pt - g'(x) = f(x)

1pt - g'' > 0

1pt - answer

So g is concave up on $(-4, 0) \cup (3, 6)$ b/c

g'' > 0 on $(-4, 0) \cup (3, 6)$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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