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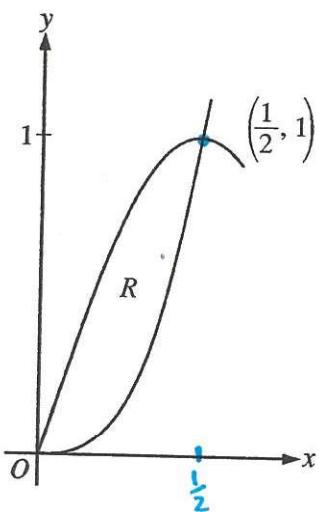
CALCULUS AB

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



3. Let R be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

- (a) Write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

$$y - y_1 = m(x - x_1)$$

$$f\left(\frac{1}{2}\right) = 8\left(\frac{1}{2}\right)^3$$

$$= 1$$

$$y - 1 = 6\left(x - \frac{1}{2}\right)$$

$$f'(x) = 24x^2$$

$$f'\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right)^2$$

$$= 6$$

1 pt - $f'\left(\frac{1}{2}\right)$
1 pt - answer

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Continue problem 3 on page 9.

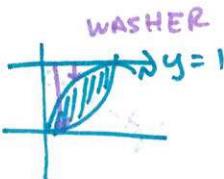
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- (b) Find the area of R .

$$\begin{aligned}
 \text{Area of } R &= \int_0^{\frac{\pi}{2}} (\sin \pi x - 8x^3) dx && \text{1 pt - integrand} \\
 &= \int_0^{\frac{\pi}{2}} \sin \pi x dx - \int_0^{\frac{\pi}{2}} 8x^3 dx \\
 &= \int_0^{\frac{\pi}{2}} \sin u \cdot \frac{du}{\pi} - 2x^4 \Big|_0^{\frac{\pi}{2}} && \text{2 pts - antiderivatives} \\
 &= \frac{1}{\pi} (-\cos u) \Big|_0^{\frac{\pi}{2}} - [2(\frac{1}{2})^4 - 0] \\
 &= -\frac{1}{\pi} (\cos \frac{\pi}{2} - \cos 0) - \frac{1}{8} && \leftarrow \text{ok to stop here} \\
 &= -\frac{1}{\pi} (0 - 1) - \frac{1}{8} \\
 &= \frac{1}{\pi} - \frac{1}{8}
 \end{aligned}$$

- (c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 1$.



$$V = \pi \int_0^{\frac{\pi}{2}} [(1-f(x))^2 - (1-g(x))^2] dx$$

1 pt - limits & constant

2 pts - integrand

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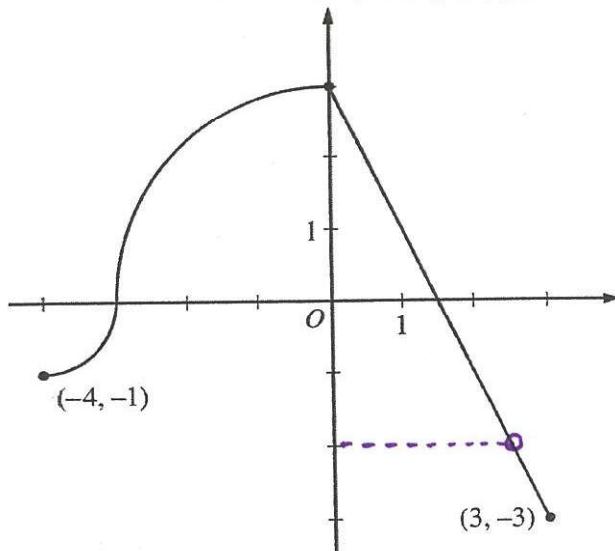
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Graph of f

4. The continuous function f is defined on the interval $-4 \leq x \leq 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t) dt$.

- (a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

$$\begin{aligned} g(-3) &= 2(-3) + \int_0^{-3} f(t) dt \\ &= -6 - \int_{-3}^0 f(t) dt \\ &= -6 - \frac{\pi}{4}(3)^2 \\ &= -6 - \frac{9\pi}{4} \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{d}{dx} \left(2x + \int_0^x f(t) dt \right) \\ g'(x) &= 2 + f(x) \\ g'(-3) &= 2 + f(-3) \\ &= 2 + 0 \\ g'(-3) &= 2 \end{aligned}$$

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- (b) Determine the x -coordinate of the point at which g has an absolute maximum on the interval $-4 \leq x \leq 3$. Justify your answer.

$$\begin{aligned} g'(x) &= 2 + f(x) \\ 0 &= 2 + f(x) \quad \rightarrow f(x) = -\frac{2}{3}x + 3 \\ f(x) &= -2 \quad \rightarrow -2 = -\frac{2}{3}x + 3 \\ -2x + 3 &= -2 \\ -2x &= -5 \\ x &= \frac{5}{2} \end{aligned}$$

$\hookrightarrow g'$ changes from + to -, check end pts

$$\begin{aligned} g\left(\frac{5}{2}\right) &= 2\left(\frac{5}{2}\right) + \int_0^{\frac{5}{2}} f(t) dt \\ &= 5 + \frac{1}{2}(1.5)(3) + \frac{1}{2}(1)(-2) = 5 + \frac{9}{4} - 1 = \frac{25}{4} \end{aligned}$$

$$\begin{aligned} g(-4) &= 2(-4) + \int_0^{-4} f(t) dt = -8 - \int_0^{-4} f(t) dt \\ &= -8 - \left(-\frac{1}{4}(1)^2 + \frac{\pi}{4}(3)^2\right) = -8 - 2\pi \end{aligned}$$

g has abs max @

$$x = \frac{5}{2}$$

$$g\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right) + \int_0^{\frac{5}{2}} f(t) dt = 6$$

1 pt
answer w/
reason

Continue problem 4 on page

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- (c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

$$g'(x) = 2 + f(x)$$

$$g''(x) = f'(x)$$

$$f'(x) = 6 \text{ or DNE}$$

$$@ x = -3, x = 0$$

$$\begin{array}{c} g'' \\ \hline -3 & 0 \end{array} \quad \begin{array}{c} + & + & + & - \\ \hline \end{array}$$

" g'' changes signs

(pt \rightarrow answer w/
reason)

g has inf pt @ $x = 0$ b/c g'' changes signs @ $x = 0$

-
- (d) Find the average rate of change of f on the interval $-4 \leq x \leq 3$. There is no point c , $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

$$\begin{aligned} \text{avg rate of change} &= \frac{f(3) - f(-4)}{3 - (-4)} \\ &= \frac{-3 - (-1)}{7} \\ &= -\frac{2}{7} \end{aligned}$$

MVT \rightarrow f cont on $[-4, 3]$,

"sharp turn"

but f is not diff'ble on $(-4, 3)$ b/c $\lim_{x \rightarrow 0^-} f'(x) \neq \lim_{x \rightarrow 0^+} f'(x)$

\therefore MVT does not guarantee a c on $(-4, 3)$ for
which $f'(c) = -\frac{2}{7}$

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5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

- (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

$$y - y_1 = m(x - x_1)$$

$$W - W_1 = \frac{dW}{dt}(t - t_1)$$

$$W - 1400 = 44(t - 0)$$

$$W - 1400 = 44\left(\frac{1}{4}\right)$$

$$W = 11 + 1400$$

$$W\left(\frac{1}{4}\right) = 1411 \text{ tons}$$

(0, 1400)

$$\begin{aligned} \left. \frac{dW}{dt} \right|_{t=0} &= \frac{1}{25}(1400 - 300) \\ &= \frac{1100}{25} \\ &= 44 \end{aligned}$$

1 pt $\rightarrow \frac{dW}{dt}|_{t=0}$

1 pt - answer

- (b) Find $\frac{d^2W}{dt^2}$ in terms of W . Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

$$\frac{dW}{dt} = \frac{1}{25}(W - 300)$$

$$\frac{d^2W}{dt^2} = \frac{1}{25} \frac{dW}{dt}$$

$$= \frac{1}{25} \left(\frac{1}{25}(W - 300) \right) \leftarrow \text{ok to stop here}$$

$$= \frac{1}{625}(W - 300)$$

1 pt - $\frac{d^2W}{dt^2}$

$$\left. \frac{d^2W}{dt^2} \right|_{t=\frac{1}{4}} > 0, \therefore \text{part (a) is an underestimate}$$

1 pt \rightarrow answer w/
reason

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- (c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

$$\frac{dW}{dt} = \frac{1}{25}(W - 300)$$

$$dW = \frac{1}{25}(W - 300) dt$$

$$\int \frac{1}{W-300} dW = \int \frac{1}{25} dt$$

$$\int \frac{1}{u} du = \frac{1}{25} t + C$$

$$\ln|u| = \frac{1}{25}t + C$$

$$\ln|W-300| = \frac{1}{25}t + C$$

$$\ln|1400-300| = \frac{1}{25}(0) + C$$

$$\ln 1100 = C$$

$$u = W - 300$$

$$\frac{du}{dw} = 1$$

$$du = dw$$

1 pt \rightarrow separate variables

1 pt \rightarrow antiderivative

1 pt \rightarrow "+C"

1 pt \rightarrow initial condition

$$\ln|W-300| = \frac{1}{25}t + \ln 1100$$

$$e^{\ln|W-300|} = e^{\frac{1}{25}t + \ln 1100}$$

$$W-300 = \pm e^{\frac{1}{25}t + \ln 1100}$$

$$W-300 = e^{\frac{1}{25}t + \ln 1100}$$

$$W = e^{\frac{1}{25}t + \ln 1100} + 300$$

(keep "+" b/c $W-300$ is positive
when $W=1400$)

\leftarrow ok to stop
here

$$\text{OR } W = e^{\frac{1}{25}t} \cdot e^{\ln 1100} + 300$$

$$W = 1100e^{\frac{1}{25}t} + 300$$

1 pt - solve for W

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6. Let f be a function defined by $f(x) = \begin{cases} 1 - 2 \sin x & \text{for } x \leq 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

- (a) Show that f is continuous at $x = 0$.

$$\hookrightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = y\text{-value}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$$f(0) = 1 - 2 \sin 0$$

$$= 1$$

$$\lim_{x \rightarrow 0^-} (1 - 2 \sin x) = \lim_{x \rightarrow 0^+} (e^{-4x})$$

$$1 - 2 \sin 0 = e^{-4(0)}$$

$$1 = 1 \therefore \lim_{x \rightarrow 0} f(x) = 1$$

2 pts \rightarrow analysis

f cont @ $x = 0$ b/c $\lim_{x \rightarrow 0} f(x) = f(0) = 1$

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- (b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

$$f'(x) = \begin{cases} -2 \cos x & x \leq 0 \\ -4e^{-4x} & x > 0 \end{cases}$$

2 pts \rightarrow $f'(x)$

$$f'(x) = -3$$

$$-2 \cos x = -3$$

$$\cos x = \frac{3}{2}$$

(cosine is ≤ 1)

$$-4e^{-4x} = -3$$

$$\ln e^{-4x} = \frac{3}{4}$$

$$-4x = \ln(\frac{3}{4})$$

$$x = \frac{\ln(\frac{3}{4})}{-4}$$

1 pt - value of x

Continue problem 6 on page 15.

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- (c) Find the average value of f on the interval $[-1, 1]$.

$$\overrightarrow{b-a} \int_a^b f$$

$$\text{avg value of } f = \frac{1}{1 - (-1)} \int_{-1}^1 f(x) dx$$

$$= \frac{1}{2} \left[\int_{-1}^0 f(u) du + \int_0^1 f(x) dx \right]$$

$$= \frac{1}{2} \left[\int_{-1}^0 (1 - 2\sin x) dx + \int_0^1 e^{-4x} dx \right]$$

$$= \frac{1}{2} \left[\int_{-1}^0 (x + 2\cos x) dx + \int_0^{-4} e^u \cdot \frac{du}{-4} \right]$$

\nearrow antiderivative

$$= \frac{1}{2} \left[0 + 2\cos 0 - (-1 + 2\cos(-1)) + -\frac{1}{4} e^u \Big|_0^{-4} \right]$$

1 pt \rightarrow correct integrals

$$u = -4x \quad u(1) = -4$$

$$\frac{du}{dx} = -4 \quad u(0) = 0$$

$$du = -4dx$$

$$\frac{du}{-4} = dx$$

$$= \frac{1}{2} \left[2\cos 0 - (-1 + 2\cos(-1)) + -\frac{1}{4} e^{-4} - (-\frac{1}{4} e^0) \right]$$

ok to
stop
here

$$= \frac{1}{2} \left[2 + 1 - 2\cos(-1) - \frac{1}{4} e^{-4} + \frac{1}{4} \right]$$

$$= \frac{1}{2} \left[\frac{13}{4} - 2\cos(-1) - \frac{1}{4} e^{-4} \right]$$

1 pt \rightarrow answer

$$= \frac{13}{8} - \cos(-1) - \frac{1}{8} e^{-4}$$

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