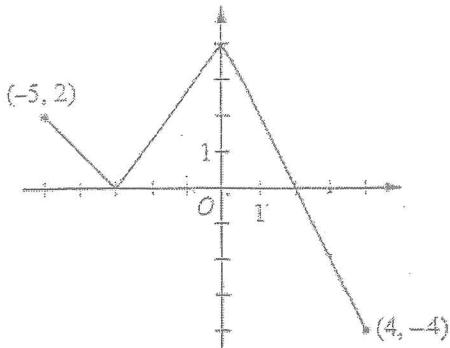


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NO CALCULATOR ALLOWED



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-5}^x f(t) dt$.

- (a) Find $g(3)$.

$$\begin{aligned} g(x) &= \int_{-3}^x f(t) dt \\ &= \frac{1}{2}(5)(4) + \frac{1}{2}(1)(-2) \\ &= 10 - 1 \\ &= 9 \end{aligned}$$

1 pt - answer

Do not write beyond this border.

- (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$$g' > 0 \quad g'' < 0$$

$$g'(x) = \frac{d}{dx} \int_{-3}^x f(t) dt$$

$$g'(x) = f(x) \quad g'(x) = 0 \rightarrow \begin{array}{c} g''(x) \\ -3 \\ + \\ 2 \end{array} \rightarrow g' > 0 \text{ on } (-5, 2)$$

$$g''(x) = f'(x) \quad f'(x) = 0 \rightarrow \begin{array}{c} g''(x) \\ -3 \\ + \\ 0 \end{array} \rightarrow g'' < 0 \text{ on } (-5, -3) \cup (0, 4)$$

g inc & conc. down on $(-5, -3) \cup (0, 2)$ b/c $g' > 0$ & $g'' < 0$
on $(-5, -3) \cup (0, 2)$

1 pt - answer
1 pt - reason

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NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$$h'(x) = \frac{5x(g'(x)) - g(x) \cdot 5}{(5x)^2} \quad 2\text{pts} - h'(x)$$

$$\begin{aligned} h'(3) &= \frac{5 \cdot 3 g'(3) - g(3) \cdot 5}{(5 \cdot 3)^2} \\ &= \frac{15(-2) - 9 \cdot 5}{(15)^2} \\ &= -\frac{1}{3} \end{aligned}$$

1 pt - answer

$g'(x) = f(x)$
 $g'(3) = f(3)$
 $g'(3) = -2$

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Do not write beyond this border.

(d) The function p is defined by $p(x) = f(\tilde{x^2 - x})$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

chain!

 $\hookrightarrow p'(x) @ x = -1$

$$p'(x) = f'(x^2 - x) \cdot (2x - 1) \quad 2\text{pts} - p'(x)$$

$$\begin{aligned} p(-1) &= f'((-1)^2 - (-1)) \cdot (2(-1) - 1) \\ &= f'(2) \cdot (-3) \\ &= -2(-3) \\ &= 6 \end{aligned}$$

1 pt - answer

$f'(2) = \text{slope}$
 $= \frac{\text{rise}}{\text{run}}$
 $= \frac{-8}{4}$
 $= -2$

4

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4

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

- (a) Find the average acceleration of train A over the interval $2 \leq t \leq 8$.

$$\begin{aligned} \text{avg acceleration} \\ \text{of Train A} &= \frac{v_A(8) - v_A(2)}{8 - 2} \\ &= \frac{-120 - 100}{6} \\ &= \frac{-220}{6} \\ &= -\frac{110}{3} \text{ meters/min/min} \\ &\text{or } -\frac{110}{3} \text{ meters/min}^2 \end{aligned}$$

meters/min ... \ominus
min

1 pt - avg. acceleration

- (b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

MVT? INT?

Train A velocity is diff'able, \therefore , velocity is cont.

Since $v_A(5) > -100$ and $v_A(8) < -100$, ← 1 pt - compare
 $v_A(5) \approx v_A(8)$
to -100

then \exists some time t on $(5, 8)$ s.t.

train A's velocity is -100 m/min.

(IVT)

1 pt - conclusion,
using
IVT

4

4

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4

NO CALCULATOR ALLOWED

- (c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

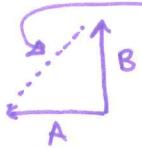
$$\begin{aligned} x_A(12) &= 300 + \int_2^{12} v_A(t) dt && \leftarrow 1 \text{ pt - position} \\ &= 300 + \left[\frac{1}{2}(100+40)(3) + \frac{1}{2}(40+(-20))(3) + \frac{1}{2}(-120+(-150))(4) \right] && \leftarrow 1 \text{ pt - trapezoid sub} \\ &= 300 + \frac{1}{2}(140 \cdot 3 + -80 \cdot 3 + -270 \cdot 4) \\ &= -150 \text{ meters} \end{aligned}$$

Train is 150 meters west of station

$\leftarrow 1 \text{ pt - position}$
 $\leftarrow t=12$

Do not write beyond this border.

- (d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$$A^2 + B^2 = C^2$$

$$\text{at } t = 2,$$

$$A = 300 \text{ (given in part c)}$$

$$B = 400$$

$$\frac{dc}{dt} = ?$$

$$2A \frac{da}{dt} + 2B \frac{db}{dt} = 2C \frac{dc}{dt}$$

$\leftarrow 2 \text{ pts - implicit differentiation of distance}$

$$2(300)(v_A(2)) + 2(400)(v_B(2)) = 2(500)(\frac{dc}{dt})$$

$$2(300)(100) + 2(400)(-5(2)^2 + 60(2) + 25) = 2(500)\frac{dc}{dt}$$

$$3000 + 4(-20 + 120 + 25) = 5\frac{dc}{dt}$$

$$300 + 4(125) = 5\frac{dc}{dt}$$

$$800 = 5\frac{dc}{dt}$$

$$\frac{dc}{dt} = 160 \text{ m/min}$$

$\leftarrow 1 \text{ pt - answer}$

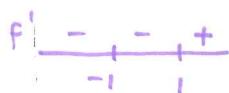
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NO CALCULATOR ALLOWED

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

5. The twice-differentiable functions f and g are defined for all real numbers x . Values of f , f' , g , and g' for various values of x are given in the table above.

- (a) Find the x -coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.
 $\rightarrow f' \text{ changes neg to pos.}$



f has rel. min @ $x=1$ b/c

1 pt - answer w/reason

f' changes from neg to pos @ $x=1$.

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- (b) Explain why there must be a value c , for $-1 < c < 1$, such that $f''(c) = 0$.

MVT? INT?

Slope of f' , ... use MVT

MVT $\rightarrow f$ is twice diff'able,

$\therefore f'$ is diff'able + cont. on $(-1, 1)$

$$f''(c) = \frac{f'(1) - f'(-1)}{1 - (-1)}$$

$$= \frac{f'(1) - f'(-1)}{2}$$

$$= \frac{0 - 0}{2}$$

$$f''(c) = 0$$

1 pt - $f'(1) - f'(-1) = 0$

1 pt - explain w/MVT

By MVT, since f' diff'able + cont on $(1, -1)$,

$\exists a, c$ on $(-1, 1)$ s.t. $f''(c) = 0$

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NO CALCULATOR ALLOWED

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chain!

- (c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

$$h'(x) = \frac{1}{f(x)} \cdot f'(x)$$

← 2 pts - $h'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3)$$

$$= \frac{1}{7} \cdot \frac{1}{2}$$

← 1 pt - answer

$$= \frac{1}{14}$$

product... so substitution

- (d) Evaluate $\int_{-2}^3 f'(g(x))g'(x) dx$.

$$u = g(x)$$

$$u(3) = g(3)$$

$$\frac{du}{dx} = g'(x)$$

$$= 1$$

$$du = g'(x) dx$$

$$u(-2) = g(-2)$$

$$=-1$$

from
table

$$\frac{du}{g'(x)} = dx$$

$$\int_{-1}^1 f'(u) \cdot g'(x) \cdot \frac{du}{g'(x)}$$

$$= \int_{-1}^1 f'(u) du$$

← 2 pts - antiderivative

$$= f(u) \Big|_{-1}^1$$

$$= f(1) - f(-1) = 2 - 8 \quad \leftarrow 1 \text{ pt - answer}$$

$$= -6$$

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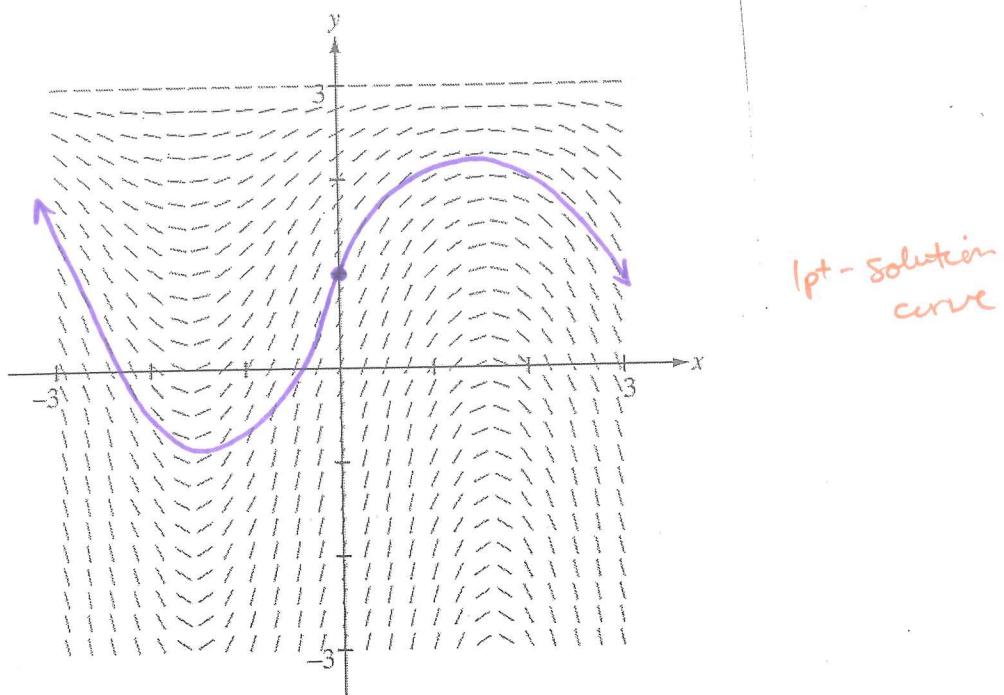
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NO CALCULATOR ALLOWED

6. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

- (a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



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- (b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

$$y - y_1 = m(x - x_1)$$

$$m \rightarrow \left. \frac{dy}{dx} \right|_{(0,1)} = (3-1)\cos 0$$

$$= 2$$

$$y - 1 = 2(x - 0) \quad \leftarrow \quad 1 \text{ pt - tangent line equation}$$

$$f(0.2) - 1 = 2(0.2 - 0)$$

$$f(0.2) = .4 + 1$$

$$= 1.4 \quad \leftarrow \quad 1 \text{ pt - approximation}$$

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NO CALCULATOR ALLOWED

- (c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

$\hookrightarrow x \text{ w/dx's, } y \text{ w/ dy's}$

$$\frac{dy}{dx} = (3-y) \cos x$$

$$dy = (3-y) \cos x \, dx$$

$$\frac{1}{3-y} \, dy = \cos x \, dx \quad \leftarrow 1\text{pt} - \text{separation variables}$$

$$\int \frac{1}{3-y} \, dy = \int \cos x \, dx$$

$$\bullet -\int \frac{1}{u} \, du = \sin x + C$$

$$-\ln|u| = \sin x + C$$

$$-\ln|3-y| = \sin x + C \quad \leftarrow 2\text{pts} - \text{antiderivatives}$$

$\leftarrow 1\text{pt} - "+C"$

$$-\ln|3-1| = \sin 0 + C \quad \leftarrow 1\text{pt} - \text{use initial condition}$$

$$-\ln 2 = C$$

$$-\ln|3-y| = \sin x - \ln 2$$

$$\ln|3-y| = -\sin x + \ln 2$$

$$|3-y| = e^{-\sin x + \ln 2}$$

$$3-y = e^{-\sin x + \ln 2}$$

$$-y = e^{-\sin x + \ln 2} - 3$$

$$y = -e^{-\sin x + \ln 2} + 3$$

$$\text{or } y = -2e^{-\sin x} + 3$$

$\leftarrow 1\text{pt} - \text{solves for } y$

Do not write beyond this border.

