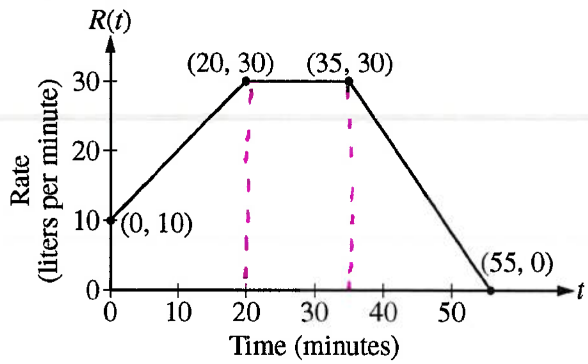


1. At time $t = 0$ minutes, a tank contains 100 liters of water. The piecewise-linear graph above shows the rate $R(t)$, in liters per minute, at which water is pumped into the tank during a 55-minute period.

(a) Find $R'(45)$. Using appropriate units, explain the meaning of your answer in the context of this problem.

$R' < 0 \rightarrow R$ dec
$R' > 0 \rightarrow R$ inc

$$R'(45) = \frac{R(55) - R(35)}{55 - 35} \quad \frac{\text{liters/min}}{\text{min}}$$

$$= \frac{0 - 30}{55 - 35}$$ ← ok to stop here w/ units written

$$= -\frac{3}{2} \text{ liters/min}^2$$

1pt: $R'(45)$

$R'(45)$ means the rate at which water is pumped into the tank, in liter/mins^2, is decreasing at $t = 45$ minutes.

1pt: explanation

(b) How many liters of water have been pumped into the tank from time $t = 0$ to time $t = 55$ minutes? Show the work that leads to your answer. → $\int$ rate

water pumped into tank $= \int_0^{55} R(t)\,dt$

$= \frac{1}{2}(10+30)(20) + 15(30) + \frac{1}{2}(20)(30)$ ← ok to stop here

$= 1150$

1pt: sum of areas
1pt: answer

Continue problem 1 on page 5.

(c) At time $t = 10$ minutes, water begins draining from the tank at a rate modeled by the function D, where $\boxed{D(t) = 10e^{(\sin t)/10}}$ liters per minute. Water continues to drain at this rate until time $t = 55$ minutes. How many liters of water are in the tank at time $t = 55$ minutes?

state

Water in tank = initial + water pumped in − water drain out

water in tank @ t=55min $= 100 + 1150 - \int_{10}^{55} D(t)\,dt$

$= 799.725$

1pt: integral
1pt: expression for water in tank
1pt: answer

(d) Using the functions R and D, determine whether the amount of water in the tank is increasing or decreasing at time $t = 45$ minutes. Justify your answer.

rate of amount >0 rate of amount <0

amount water = initial + $\int R(t)\,dt - \int D(t)\,dt$

rate of amount of water $= R(t) - D(t)$

$R(45) - D(45) = 5.888$
> 0

Amount of water in tank is increasing @ $t = 45$

b/c $R(45) - D(45) > 0$.

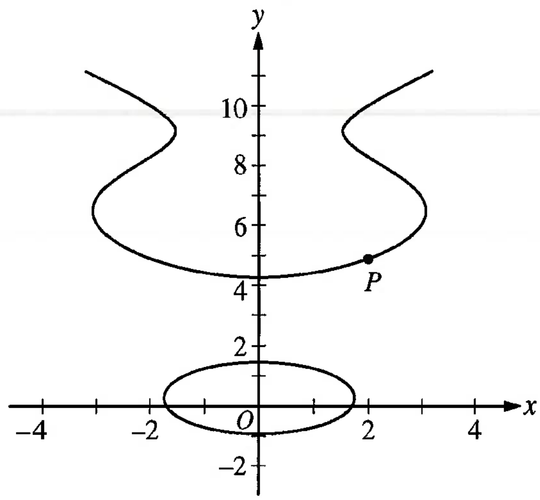
2pts: answer w/ reason

Do not write beyond this border.

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

GO ON TO THE NEXT PAGE.



2. The graph of the equation $x^2 = -2 + y + 5\cos y$ is shown above for $y \leq 11$. It is known that $\frac{dy}{dx} = \frac{2x}{1 - 5\sin y}$.

The x-coordinate of point P shown on the graph is 2. $\rightarrow (x, y) \rightarrow (2, ?)$

(a) Write an equation for the line tangent to the graph at point P.

$\hookrightarrow y - y_1 = m(x - x_1)$

$x^2 = -2 + y + 5\cos y$

1pt: y-coordinate

$2^2 = -2 + y + 5\cos y$

$y = 4.928$

$\boxed{y - 4.928 = 0.680(x - 2)}$

$\left.\frac{dy}{dx}\right|_{(2, 4.928)} = 0.680$

1pt: slope

1pt: tangent line eq.

Continue problem 2 on page 7.

Do not write beyond this border.

(b) For $y \le 11$, find the y-coordinate of each point on the graph where the line tangent to the graph at that point is vertical.

↳ $\frac{dy}{dx}$ DNE when line vertical

DNE when denom = 0

$$\frac{dy}{dx} = \frac{2x}{1-5\sin y}$$

$1 - 5\sin y = 0$

$y = 0.201$, $y = 6.485$, $y = 9.223$

1pt: set $1-5\sin y = 0$

2pt: y-coordinates

(c) Find the average value of the x-coordinates of the points on the graph in the first quadrant between $y = 5$ and $y = 9$.

↳ $\frac{1}{b-a}\int_a^b$ x-coordinate

↳ x is positive

$x^2 = -2 + y + 5\cos y$

$x = \sqrt{-2+y+5\cos y}$

± ... x in QI, so $x > 0$

$$\text{Avg value of x-coord.} = \frac{1}{9-5}\int_5^9 \sqrt{-2+y+5\cos y}\, dy$$

$= 2.550$

1pt: integrand

1pt: limits + constant

1pt: answer.

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

$\Delta t = 3$ $\Delta t = 2$ $\Delta t = 3$ $\Delta t = 4$

t (seconds)	0	3	5	8	12
$k(t)$ (feet per second)	0	5 R	10 R	20 R	24 R

3. Kathleen skates on a straight track. She starts from rest at the starting line at time $t = 0$. For $0 < t \le 12$ seconds, Kathleen's velocity k, measured in feet per second, is differentiable and increasing. Values of $k(t)$ at various times t are given in the table above.

$a(t) = v'(t)$

(a) Use the data in the table to estimate Kathleen's acceleration at time $t = 4$ seconds. Show the computations that lead to your answer. Indicate units of measure.

$$a(4) = v'(4)$$
$$= \frac{k(5) - k(3)}{5-3} \quad \begin{matrix} \to \text{ft/sec} \\ \to \text{sec} \end{matrix}$$
$$= \frac{10-5}{5-3}$$
$$= \frac{5}{2} \text{ ft/sec}^2$$

1pt: estimate
1pt: units

(b) Use a right Riemann sum with the four subintervals indicated by the data in the table to approximate $\boxed{\int_0^{12} k(t)\,dt}$. Indicate units of measure. Is this approximation an overestimate or an underestimate for the value of $\int_0^{12} k(t)\,dt$? Explain your reasoning.

$(\text{sec})\left(\frac{\text{ft}}{\text{sec}}\right) = \text{ft}$

undr, dec
over, inc

$$\int_0^{12} k(t)\,dt \approx 4(24) + 3(20) + 2(10) + 3(5)$$
$$= 191 \text{ ft}$$

1pt: Right Rieman sum
1pt: approx w/ units

This approx is an overestimate of $\int_0^{12} k(t)\,dt$ b/c k is inc on $[0, 12]$.

1pt: overestimate w/ reason

Do not write beyond this border. Do not write beyond this border.

Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

(c) Nathan skates on the same track, starting 5 feet ahead of Kathleen at time $t = 0$. Nathan's velocity, in feet per second, is given by $n(t) = \dfrac{150}{t+3} - 50e^{-t}$. Write, but do not evaluate, an expression involving an integral that gives Nathan's distance from the starting line at time $t = 12$ seconds.

↳ position from starting line (since Nathan won't skate back)

$$\text{Nathan @ } t=12 = \text{initial position} + \int_0^{12} n(t)\,dt$$

$$= 5 + \int_0^{12} n(t)\,dt$$

1pt: integral

1pt: answer

(d) Write an expression for Nathan's acceleration in terms of t.

↳ $a(t) = v'(t)$

nathan's velocity $= n(t)$

$$\text{nathan's acceleration} = n'(t)$$

$$= \frac{(t+3)(0) - 150(1)}{(t+3)^2} - 50(-e^{-t})$$

2pts: $n'(t)$

OR

$$n(t) = 150(t+3)^{-1} - 50e^{-t}$$

$$\text{nathan's acceleration} = -150(t+3)^{-2} - 50(-e^{-t})$$

Do not write beyond this border.

Do not write beyond this border.

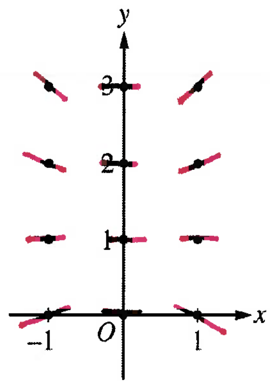
GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = \frac{x(y-1)}{4}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.

$y \backslash x$	-1	0	1
3	$-1/2$	0	$1/2$
2	$-1/4$	0	$1/4$
1	0	0	0
0	$1/4$	0	$-1/4$



1pt: zero slopes
1pt: other slopes

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 3$. Write an equation for the line tangent to the graph of f at the point $(1, 3)$ and use it to approximate $f(1.4)$.

↳ $y - y_1 = m(x - x_1)$ $\qquad (x_1, y_1)$ $\qquad$ ↳ $f(1.4) \approx ?$

$y - 3 = \frac{1}{2}(x - 1)$ $\qquad \left.\frac{dy}{dx}\right|_{(1,3)} = \frac{1}{2}$ (from table above)

$y - 3 = \frac{1}{2}(1.4 - 1)$

$y = \frac{1}{2}(1.4 - 1) + 3$

$f(1.4) \approx \frac{1}{2}(1.4 - 1) + 3$

OR

$f(1.4) \approx 3.2$

1pt: tangent line

1pt: approx.

Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = 3$.

→ x w/ dx, y w/ dy

$$\frac{dy}{dx} = \frac{x(y-1)}{4}$$

$$\int \frac{1}{y-1}\,dy = \int \frac{x}{4}\,dx$$

1pt: separate variables

$u = y-1$

$du = dy$

$$\int \frac{1}{u}\,du = \frac{1}{4}\int x\,dx$$

$$\ln|u| = \frac{1}{4}\cdot\frac{1}{2}x^2 + C$$

2pts: antiderivatives

$$\ln|y-1| = \frac{1}{8}x^2 + C$$

initial condition (1,3)

$$\ln|3-1| = \frac{1}{8} + C$$

$$\ln 2 = \frac{1}{8} + C$$

$$\ln 2 - \frac{1}{8} = C$$

1pt: "+C" and initial condition

$$e^{\ln|y-1|} = e^{\frac{1}{8}x^2 + \ln 2 - \frac{1}{8}}$$

$$|y-1| = e^{\frac{1}{8}x^2 + \ln 2 - \frac{1}{8}}$$

pos b/c $y-1>0$, $3-1>0$, $2>0$

$$y-1 = e^{\frac{1}{8}x^2 + \ln 2 - \frac{1}{8}}$$

$$y = e^{\frac{1}{8}x^2 + \ln 2 - \frac{1}{8}} + 1$$

1pt: solve for y

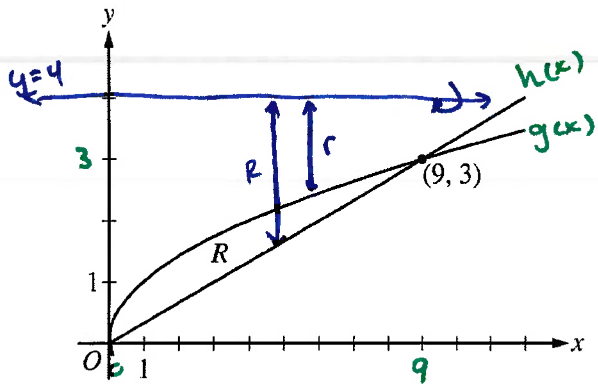
or

$$y = 2e^{\frac{1}{8}x^2 - \frac{1}{8}} + 1$$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.



5. Let R be the region in the first quadrant enclosed by the graphs of $g(x) = \sqrt{x}$ and $h(x) = \frac{x}{3}$, as shown in the figure above.

(a) Find the area of region R.

Area of R $= \int_0^9 (\sqrt{x} - \frac{x}{3})\,dx$

1pt: integrand
1pt: ~~[illegible]~~ antiderivative

$= \int_0^9 (x^{1/2} - \frac{1}{3}x)\,dx$

$= (\frac{2}{3}x^{3/2} - \frac{1}{6}x^2)\Big|_0^9$

$= \frac{2}{3}(9)^{3/2} - \frac{1}{6}(9)^2 - 0$ ← ok to stop here

1pt: answer

$= \frac{2}{3}(\sqrt{9})^3 - \frac{1}{6}\cdot 81$

$= \frac{2}{3}\cdot 27 - \frac{27}{2}$

$= 18 - \frac{27}{2}$

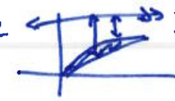
$= \frac{36}{2} - \frac{27}{2}$

$= \frac{9}{2}$

Continue problem 5 on page 19.

(b) Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when R is revolved about the horizontal line $y = 4$.

↳ WASHER



$$\text{Volume} = \pi \int_0^9 \left[\left(4 - \cancel{g(x)}\, h(x)\right)^2 - \left(4 - g(x)\right)^2 \right] dy$$

2pts: integrand
1pt: limits & constant

(c) Find the maximum vertical distance between the graph of g and the graph of h between $x = 0$ and $x = 16$. Justify your answer.

→ abs max. → crit #s + endpts.

↳ difference in y-values btn $g(x)$ + $h(x)$

$\text{distance} = |g(x) - h(x)|$

Let, $D(x) = g(x) - h(x)$

$= x^{1/2} - \frac{1}{3}x$

$D'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3}$

1pt: set $D' = 0$

$0 = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$\frac{1}{3} = \frac{1}{2\sqrt{x}}$

$2\sqrt{x} = 3$

$\sqrt{x} = \frac{3}{2}$

$x = \frac{9}{4}$

1pt: crit # $\frac{9}{4}$

1pt: answer w/reason

evaluate endpts $\begin{cases} D(0) = g(0) - h(0) = 0 \\ |D(16)| = |g(16) - h(16)| = |-4/3| = 4/3 \end{cases}$ → max vertical distance is $\frac{4}{3}$

$|D(9/4)| = |g(9/4) - h(9/4)| = |3/4| = 3/4$

Do not write beyond this border.

Do not write beyond this border.

NO CALCULATOR ALLOWED

6. Let $g(x) = 4(x+1)^{-2/3}$ and let f be the function defined by $f(x) = \int_0^x g(t)\,dt$ for $x \geq 0$.

(a) Find $f(26)$.

$f(26) = \int_0^{26} g(t)\,dt$

$= \int_0^{26} 4(x+1)^{-2/3}\,dx$ $\quad u = x+1 \to u(26) = 27$, $u(0) = 1$; $du = dx$

$= 4\int_1^{27} u^{-2/3}\,du$ $\quad$ 2pts: antiderivative

$= 4(3u^{1/3})\Big|_1^{27}$ $\quad$ 1pt: answer

$= 12(27^{1/3} - 1^{1/3})$ ← ok to stop here.

$= 12(\sqrt[3]{27} - 1)$

$= 12(3-1) = 24$

(b) Determine the $\underset{\hookrightarrow f''}{\underline{\text{concavity}}}$ of the graph of $y = f(x)$ for $\underset{\hookrightarrow (0,\infty)}{\underline{x > 0}}$. Justify your answer.

$f(x) = \int_0^x g(t)\,dt$

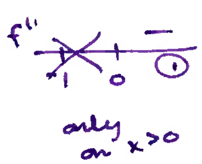
$f'(x) = g(x)$ $\quad$ 1pt: $f'(x)$

$f''(x) = g'(x)$ $\quad$ 1pt: $f''(x)$

$= 4\left(-\frac{2}{3}(x+1)^{-5/3}\right)$

$= \frac{-8}{3} \cdot \frac{1}{\sqrt[5]{(x+1)^3}}$ $\quad$ 1pt: answer w/ reason

$f''(x)$ DNE @ $x = -1$



f is concave down on $(0,\infty)$ b/c $f'' < 0$ on $(0,\infty)$

Do not write beyond this border.

Do not write beyond this border.

Continue problem 6 on page 21.

(c) Let h be the function defined by $h(x) = x - f(x)$. Find the minimum value of h on the interval $0 \le x \le 26$.

↳ abs min, crit#s + endpts.

$h(x) = x - f(x)$

$h'(x) = 1 - f'(x)$

$= 1 - 4(x+1)^{-2/3}$

1pt: sets $h' = 0$

$h'(x) = 0$

$0 = 1 - \dfrac{4}{\sqrt[3]{(x+1)^2}}$

$\dfrac{4}{\sqrt[3]{(x+1)^2}} = 1$

$4 = \sqrt[3]{(x+1)^2}$

$4^3 = (x+1)^2$

$\sqrt{4^3} = x+1$

$8 = x+1$

$7 = x$

1pt: $x=7$ crit #

evaluate endpts

$h(0) = 0 - f(0) = 0$ part (a)

$h(26) = 26 - f(26) = 26 - 24$

$= 2$

evaluate crit#

$h(7) = 7 - f(7)$

$= 7 - \int_0^7 g(x)\,dx$

$u(7) = 8$

$u(0) = 1$

From part (a)

$= 7 - 4(3u^{1/3})\Big|_1^8$

1pt: answer w/ reason

$= 7 - 12(8^{1/3} - 1^{1/3})$

$= 7 - 12(2-1)$

$= 7 - 12 = -5$

abs. min value is -5.

GO ON TO THE NEXT PAGE.