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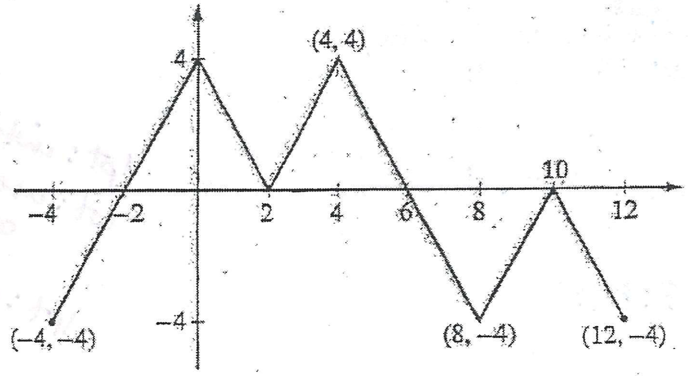

Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

↳ g' pos, neg ↳ g' neg, pos

$g(x) = \int_2^x f(t)\,dt$

$g'(x) = f(x)$

g' − | − (10)

1pt: answer w/ reason

g has neither @ x = 10 b/c g' does not change signs @ x = 10.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

↳ g'' changes signs

$g''(x) = f'(x)$

g'' + | − (4)

1pt: answer w/ reason

g has inf. pt @ x = 4 b/c g'' changes signs @ x = 4

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Continue problem 3 on page 15.

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(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers. → check rel. min, rel max and endpts.

$g' = f$

[− | + | + | − | −]
−4, −2, 2, 6, 10, 12

→ g has rel. max @ x=6 b/c g' changes from pos to neg @ x=6

1pt: considers x=−2 and x=6

g has rel. min @ x=−2 b/c g' changes from neg to pos @ x=−2

crit #s

$g(-2) = \int_2^{-2} f(t)\,dt$

$= -\int_{-2}^{2} f(t)\,dt = -\frac{1}{2}(4)(4) = -8$

$g(6) = \int_2^6 f(t)\,dt = \frac{1}{2}(4)(4) = 8$

endpts

$g(-4) = \int_2^{-4} f(t)\,dt$

$= -\int_{-4}^{2} f(t)\,dt = -\left[-\frac{1}{2}(2)(4) + 8\right]$

$= -[4] = -4$

$g(12) = \int_2^{12} f(t)\,dt = -\frac{1}{2}(2)(4) = -4$

1pt: considers x=−4 and x=12 (endpts)

2pts: answer w/ justification

abs max value is 8, abs min value is −8.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$g(x) \le 0$ on $[-4, 2] \cup [10, 12]$

2 pts: intervals

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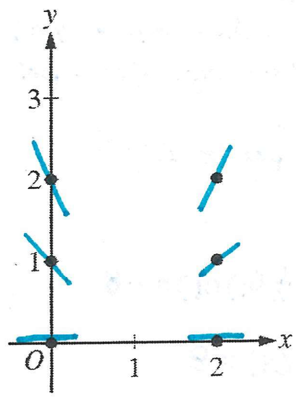
4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

y \ x	0	2
2	-4	4
1	-1	1
0	0	0



1pt: zero slopes
1pt: non-zero slopes

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$y - y_1 = m(x - x_1)$

$y - 3 = 9(x - 2)$

$y - 3 = 9(2.1 - 2)$

$y = 9(2.1 - 2) + 3$

$f(2.1) \approx 3.9$

1pt: tangent line equation

1pt: approximation

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Continue problem 4 on page 17.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $\boxed{f(2) = 3}$.

↳ x w/ dx , y w/ dy

(2, 3)

$\frac{dy}{dx} = \frac{y^2}{x-1}$ @ (2,3)

$dy = \frac{y^2}{x-1}\,dx$

$\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$

1pt: separation of variables

$u = x-1$

$du = dx$

$\int y^{-2}\,dy = \int \frac{1}{u}\cdot du$

$-y^{-1} = \ln|u| + c$

$-y^{-1} = \ln|x-1| + c$

2 pt: antiderivatives

1pt: constant "+c" and initial condition

$-3^{-1} = \ln|2-1| + c$

$-\frac{1}{3} = \ln 1 + c$

$-\frac{1}{3} = c$

$-y^{-1} = \ln|x-1| - \frac{1}{3}$

$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$

$\frac{1}{y} = -\ln|x-1| + \frac{1}{3}$

$y = \frac{1}{-\ln|x-1| + \frac{1}{3}}$

1pt: solves for y

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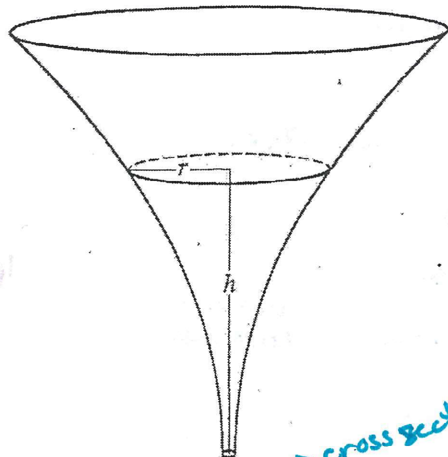

cross sections for volume!

5. The inside of a funnel of height 10 inches has $\boxed{\text{circular cross sections}}$, as shown in the figure above. At height h, the radius of the funnel is given by $\boxed{r = \frac{1}{20}\left(3 + h^2\right)}$, where $\underline{0 \le h \le 10}$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

$\hookrightarrow \frac{1}{b-a}\int_a^b R$

$$\text{average value} = \frac{1}{10-0}\int_0^{10} r(h)\,dh \qquad \text{1pt: integral}$$

$$= \frac{1}{10}\int_0^{10}\frac{1}{20}(3+h^2)\,dh$$

$$= \frac{1}{200}\int_0^{10}(3+h^2)\,dh$$

$$= \frac{1}{200}\left(3h + \tfrac{1}{3}h^3\right)\Big|_0^{10} \qquad \text{1pt: antiderivative}$$

$$= \frac{1}{200}\left(3\cdot 10 + \tfrac{1}{3}(10)^3 - 0\right)$$

$$= \frac{1}{200}\left(30 + \frac{1000}{3}\right)$$

1pt: answer

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Continue problem 5 on page 19.

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(b) Find the volume of the funnel.

Area of cross section $= \pi r^2$
$= \pi (r(h))^2$

$$\text{Volume} = \int_0^{10} \pi \left(\tfrac{1}{20}(3+h^2)\right)^2 dh$$ 1pt: integrand

$$= \left(\tfrac{1}{20}\right)^2 \pi \int_0^{10} (3+h^2)^2\, dh$$

$$= \left(\tfrac{1}{20}\right)^2 \pi \int_0^{10} (9+6h^2+h^4)\, dh$$

$$= \left(\tfrac{1}{20}\right)^2 \pi \left(9h + 2h^3 + \tfrac{1}{5}h^5\right)\Big|_0^{10}$$ 1pt: antidrivative

$$= \left(\tfrac{1}{20}\right)^2 \pi \left(9\cdot 10 + 2(10)^3 + \tfrac{1}{5}(10)^5 - 0\right)$$ 1pt: answer

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(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $\boxed{h = 3}$ inches, the radius of the surface of the liquid is decreasing at a $\boxed{\text{rate of } \frac{1}{5} \text{ inch per second}}$. At this instant, what is the $\boxed{\text{rate of change of the height}}$ of the liquid with respect to time?

$h=3$ $\quad \frac{dr}{dt} = -\frac{1}{5}$ inch/sec $\quad \frac{dh}{dt} = ?$

$$r = \tfrac{1}{20}(3+h^2)$$

$$\frac{dr}{dt} = \frac{1}{20}\left(0 + 2h\frac{dh}{dt}\right)$$ 2pts: chain rule

$$-\frac{1}{5} = \frac{1}{20}\left(2\cdot 3\cdot \frac{dh}{dt}\right)$$

$$-4 = 6\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\tfrac{2}{3} \text{ in/sec}$$ 1pt: answer

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x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	–6	3	2	8
2	2	–2	–3	0
3	8	7	6	2
6	4	5	3	–1

6. The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

$y - y_1 = m(x - x_1)$

$k(3) = f(g(3))$
$= f(6)$
$= 4$

$k'(x) = g'(x) \cdot f'(g(x))$

$k'(3) = g'(3) \cdot f'(g(3))$
$= 2 \cdot f'(6)$
$= 2 \cdot 5$
$= 10$

2pts: slope @ $x=3$

$y - 4 = 10(x - 3)$

1pt: tangent line equation

~~$k(x) - 4 = 10(x-3)$~~

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Continue problem 6 on page 21.

(b) Let $h(x) = \frac{g(x)}{f(x)}$. Find $h'(1)$.

$$h'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{(f(x))^2}$$

$$h'(1) = \frac{f(1)g'(1) - g(1)f'(1)}{(f(1))^2}$$

2pts: expression for $h'(1)$

$$= \frac{-6 \cdot 8 - 2 \cdot 3}{(-6)^2}$$

1pt: answer

$$= -\frac{3}{2}$$

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

$u = 2x$ $\quad u(3) = 6$

$du = \cancel{2}dx$ $\quad u(1) = 2$

$\frac{1}{2}du = dx$

$$\int_1^3 f''(2x)\,dx$$

$$= \int_2^6 f''(u) \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}\int_2^6 f''(u)\,du$$

$$= \frac{1}{2} f'(u) \Big|_2^6$$

2 pts: antiderivative

$$= \frac{1}{2}\left[f'(6) - f'(2)\right]$$

$$= \frac{1}{2}\left[5 - -2\right]$$

1pt: answer

$$= \frac{7}{2}$$

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