$\Delta t = 2$ $\Delta t = 3$ $\Delta t = 2$ $\Delta t = 3$

t (minutes)	0	2	5	7	10
$h(t)$ (inches)	3.5	10.0	15.5	18.5	20.0

R R R R

1. The depth of water in tank A, in inches, is modeled by a differentiable and increasing function h for $0 \le t \le 10$, where t is measured in minutes. Values of $h(t)$ for selected values of t are given in the table above.

(a) Use the data in the table to find an approximation for $h'(6)$. Show the computations that lead to your answer. Indicate units of measure.

$$h'(6) \approx \frac{h(7) - h(5)}{7 - 5}$$

inches / min

$$\approx \frac{18.5 - 15.5}{7 - 5}$$

$$= \frac{3}{2} \text{ inches/min}$$

1 pt: answer w/ units

(b) Approximate the value of $\int_0^{10} h(t)\,dt$ using a right Riemann sum with the four subintervals indicated by the data in the table. Is this approximation greater than or less than $\int_0^{10} h(t)\,dt$? Give a reason for your answer.

over → inc
under → dec

$$\int_0^{10} h(t)\,dt \approx 20(3) + 18.5(2) + 15.5(3) + 10.0(2)$$

$$= 163.5$$

1 pt: Right sum

1 pt: approx

The approx is greater than $\int_0^{10} h(t)\,db$

b/c $h(t)$ is inc on $[0, 10]$.

1 pt: overestimate w/ reason

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(c) The depth of water in tank B, in inches, is modeled by the function $\boxed{g(t) = 3.2 + 17.5\sqrt{\sin(0.16t)}}$ for $0 \le t \le 10$, where t is measured in minutes. Find the average depth of the water in tank B over the interval $0 \le t \le 10$. Is this value greater than or less than the average depth of the water in tank A over the interval $0 \le t \le 10$? Give a reason for your answer.

avg value $= \frac{1}{b-a}\int_a^b$ function

tank A?

Tank B

Avg depth of water $= \frac{1}{10-0}\int_0^{10} g(t)\,dt$

$= 16.624$

1pt: integral

1pt: avg depth for tank B

from part(b)

$16.624 > 16.35$

Tank A

Avg depth of water $= \frac{1}{10-0}\int_0^{10} h(t)\,dt$

$= 16.35$

1pt: answer w/ reason

Avg depth of water in Tank B is greater than avg ~~tank~~ depth of water in tank A.

(d) According to the model given in part (c), is the depth of the water in tank B increasing or decreasing at time $t = 6$? Give a reason for your answer.

rate positive or neg?

$g'(6) = 0.887$

1pt: $g'(6)$

1pt: answer w/reason

The depth of water in tank B is inc @ $t=6$

b/c $g'(6) > 0$.

2. Particle Q moves along the x-axis so that its velocity at any time t is given by $v_Q(t) = 1 - 3\cos\left(\frac{t^2}{5}\right)$, and its acceleration at any time t is given by $a_Q(t) = \frac{6t}{5}\sin\left(\frac{t^2}{5}\right)$. The particle is at position $\boxed{x = 2 \text{ at time } t = 0.}$

(a) In the interval $0 < t < 5$, when is the velocity of particle Q increasing? Give a reason for your answer.

$a(t) = v'(t)$

$\hookrightarrow v(t)$ inc $\rightarrow v'(t) > 0 \rightarrow a(t) > 0$

use graph calc. ☺

Velocity of particle Q is inc on $(0, 3.936)$

1pt: interval

b/c $v_Q'(t) > 0$ on $(0, 3.936)$

1pt: reason

OR $a_Q(t) > 0$ on $(0, 3.936)$

(b) Find the position of particle Q at time $t = 3$.

$\hookrightarrow x_Q(3) =$ initial position $+ \int v_Q(t)$

1pt: integral

1pt: used $x_Q(0)$

1pt: answer

$$x_Q(3) = 2 + \int_0^3 v_Q(t)\,dt$$

$$= -1.490$$

(c) A second particle, R, moves along the x-axis so that its position at any time t is given by a differentiable function $x_R(t)$, where $x_R(1) = 4$ and $x_R(3) = 8$. Explain why there must be a time t, for $1 < t < 3$, at which the velocity of particle R is 2.

↳ $v(t) = \frac{\Delta x}{\Delta t}$ MVT!

$x_R(t)$ is diff'able (given)

$x_R(t)$ is cont on $[1,3]$ b/c $x_R(t)$ is diff'able

$$v_R(t) = \frac{x_R(3) - x_R(1)}{3-1}$$

$$= \frac{8-4}{2}$$

$$v_R(t) = 2$$

1pt: $\frac{x_R(3) - x_R(1)}{3-1}$

1pt: conclusion using MVT

∴, by MVT, there must be a time, t, on $(1,3)$ at which velocity of R is 2.

(d) At time $t = 3$, the velocity of particle R described in part (c) is $\boxed{-2}$. Are particles Q and R moving toward each other or away from each other at time $t = 3$? Explain your reasoning.

velocity

Particle Q

$v_Q(3) = 1.682$
> 0

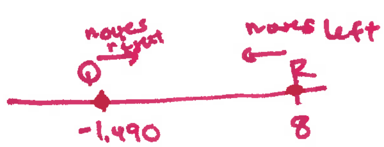
So, particle Q moves right from position,

$x_Q(3) = -1.490$

Particle R

$v_R(3) = -2$
< 0

So, particle R moves left.
from position,
$x_R(3) = 8$



1pt: answer
1pt: reason

∴, The particles are moving toward each other.

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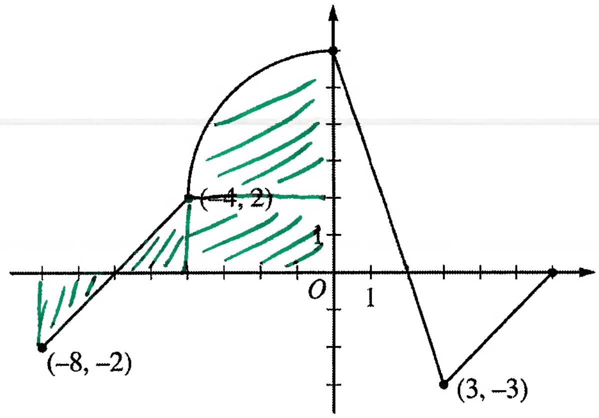

Graph of g

3. A continuous function g is defined on the closed interval $-8 \le x \le 6$. The graph of g, shown above, consists of three line segments and a quarter of a circle centered at the point $(0, 2)$. Let f be the function given by $f(x) = \int_{-8}^{x} g(t)\,dt$.

(a) Find all values of x in the interval $-8 < x < 6$ at which f has a critical point. Classify each critical point as the location of a local minimum, a local maximum, or neither. Justify your answers.

↳ f' neg to pos ↳ f' pos to neg

$f(x) = \int_{-8}^{x} g(t)\,dt$

$f'(x) = g(x)$

1pt: $f'(x) = g(x)$

g: $-$ | -6 | $+$ | 2 | $-$ | 6

crit #s $\left\{ \begin{array}{l} g(x) = 0 \\ @\ x = -6,\ 2,\ 6 \end{array} \right.$

6 ↓ not in interval $(-8, 6)$

1 pt: crit #s

f has rel. min @ $x = -6$ b/c f' changes from neg to pos @ $x = -6$

1pt: answers w/ reason

f has rel. max @ $x = 2$ b/c f' changes from pos to neg @ $x = 2$

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NO CALCULATOR ALLOWED

(b) Find $f(0)$.

$f(0) = \int_{-8}^{0} g(t)\,dt$

$= 4(2) + \frac{1}{4}\pi(4)^2$

$= 8 + 4\pi$

2 ΔS cancelled out

1pt: answer

(c) Find $\lim\limits_{x \to -4} \dfrac{f(x)}{x^2 + 4x}$.

$\lim\limits_{x\to -4} f(x) = 0$

$\lim\limits_{x\to -4} (x^2+4x) = 0$

$\lim\limits_{x\to -4} \dfrac{f(x)}{x^2+4x}$ $\to 0$, $\to 0$

$f'(-4) = g(-4)$

By L'Hôpital's rule,

$\lim\limits_{x\to -4} \dfrac{f(x)}{x^2+4x} = \lim\limits_{x\to -4} \dfrac{f'(x)}{2x+4} = \dfrac{f'(-4)}{2(-4)+4}$

$= \dfrac{2}{-4}$

$= -\dfrac{1}{2}$

1pt: used L'Hôpital correctly

1pt: answer

(d) Let h be the function defined by $h(x) = \dfrac{g(x)}{x^2 + 1}$. Find $h'(1)$.

quotient

$h'(x) = \dfrac{(x^2+1)g'(x) - g(x)(2x)}{(x^2+1)^2}$

$h'(1) = \dfrac{(1^2+1)g'(1) - g(1)(2\cdot 1)}{(1^2+1)^2}$

$= \dfrac{2(-3) - 3(2)}{(2)^2}$ ← ok to stop here

$= -3$

$g'(1) = \text{slope}$

$= \dfrac{-3-6}{3-0}$

$= -3$

$g(1) = 3$

2pts: $h'(x)$

1pt: answer

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4. Consider the differential equation $\frac{dy}{dx} = (y - 2)(x^2 + 1)$.

(a) Find $y = g(x)$, the particular solution to the given differential equation with initial condition $g(0) = 5$.

→ x w/ dx , y w/ dy

$\int \frac{1}{y-2}\,dy = \int (x^2+1)\,dx$ 1pt: separate variables

$u = y-2$
$du = dy$

$\int \frac{1}{u}\,du = \frac{1}{3}x^3 + x + C$

$\ln|u| = \frac{1}{3}x^3 + x + C$ 2pts: antiderivatives

$\ln|y-2| = \frac{1}{3}x^3 + x + C$ $\xrightarrow{\text{initial cond. }(0,5)}$ $\ln|y-2| = \frac{1}{3}x^3 + x + \ln 3$

$\ln|5-2| = C$

$\ln 3 = C$

$|y-2| = e^{\frac{1}{3}x^3 + x + \ln 3}$

1pt: "+C" and initial condition

pos. b/c
$y-2>0$
$5-2>0$
$3>0$

$y - 2 = e^{\frac{1}{3}x^3 + x + \ln 3}$

$y = e^{\frac{1}{3}x^3 + x + \ln 3} + 2$

1pt: solves for y

or

$y = 3e^{\frac{1}{3}x^3 + x} + 2$

(b) For the particular solution $y = g(x)$ found in part (a), find $\lim_{x \to -\infty} g(x)$.

$\lim_{x \to -\infty} \left(3e^{\frac{1}{3}x^3 + x} + 2\right)$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$= 0 + 2$

$= \boxed{2}$

1pt: answer

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(c) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 3$. Find the value of $\frac{d^2y}{dx^2}$ at the point $(1, 3)$. Is the graph of $y = f(x)$ concave up or concave down at the point $(1, 3)$? Give a reason for your answer.

product rule

$$\frac{dy}{dx} = (y-2)(x^2+1) \longrightarrow \left.\frac{dy}{dx}\right|_{(1,3)} = (3-2)(1^2+1) = 2$$

$$\frac{d^2y}{dx^2} = (x^2+1)\frac{dy}{dx} + (y-2)(2x)$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,3)} = (1^2+1)\left(\left.\frac{dy}{dx}\right|_{(1,3)}\right) + (3-2)(2\cdot 1)$$

2pts: $\left.\frac{d^2y}{dx^2}\right|_{(1,3)}$

$$= 2(2) + 1(2)$$

$$= 6$$

Graph of $y=f(x)$ is concave up @ $(1,3)$ b/c $\left.\frac{d^2y}{dx^2}\right|_{(1,3)} > 0$

1pt: concave up w/ reason

5. The function f is defined by

$$f(x) = \begin{cases} 3x^2 + 2x & \text{for } x \le 0 \\ e^{2x} + 2 & \text{for } x > 0. \end{cases}$$

(a) Is f continuous at $x = 0$? Justify your answer.

not required for this problem

① y-value
② limits
③ y-value = limits

$f(0) = 3(0)^2 + 2(0) = 0$

$\lim_{x\to 0^-} (3x^2+2x) \ne \lim_{x\to 0^+} (e^{2x}+2)$

1pt: considers one-sided limits

$0 \ne e^0 + 2$

$0 \ne 1+2$

$0 \ne 3$, $\therefore \lim_{x\to 0} f(x)$ DNE

1pt: answer w/ reason

So, f is not cont @ $x = 0$.

(b) Find $f'(-2)$ and $f'(3)$.

$$f'(x) = \begin{cases} 6x + 2 & x \le 0 \\ 2e^{2x} & x > 0 \end{cases}$$

$f'(-2) = 6(-2) + 2$
$= -10$

$f'(3) = 2e^{2(3)}$
$= 2e^6$

1pt: $f'(-2)$

1pt: $f'(3)$

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NO CALCULATOR ALLOWED

(c) Explain why $f'(0)$ does not exist.

f is not cont, ∴ f' not diff'able

~~$\lim_{x\to 0} f'(x)$~~

$f'(0)$ does not exist b/c f is not cont @ $x=0$

∴ proved in part (a)

1pt: explanation

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(d) Let g be the function given by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find $g(1)$.

$g(1) = \int_{-1}^{1} f(t)\,dt$

$= \int_{-1}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt$

$= \int_{-1}^{0} (3x^2+2x)\,dx + \int_{0}^{1} (e^{2x}+2)\,dx$

$= (x^3+x^2)\Big|_{-1}^{0} + \int_{0}^{1} e^{2x}\,dx + 2x\Big|_{0}^{1}$

$= 0-[(-1)^3+(-1)^2] + \frac{1}{2}\int_{0}^{2} e^{u}\,du + 2-0$

$= 1-1+\frac{1}{2}e^{u}\Big|_{0}^{2} + 2$

$= \frac{1}{2}e^2 - \frac{1}{2}e^0 + 2$ ← ok to stop here

$= \frac{1}{2}e^2 - \frac{1}{2} + 2$

$= \frac{1}{2}e^2 + \frac{3}{2}$

$u=2x$, $u(1)=2$, $u(0)=0$

$du = 2\,dx$

$\frac{1}{2}du = dx$

1pt: integrals

2pts: antiderivatives

1pt: answer

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NO CALCULATOR ALLOWED

6. A hive contains $\boxed{35}$ hundred bees at time $t = 0$. During the time interval $0 \le t \le 4$ hours, bees enter the hive at a $\boxed{\text{rate}}$ modeled by $\boxed{E(t) = 16t - 3t^2}$, where $E(t)$ is measured in hundreds of bees per hour. During the same time interval, bees leave the hive at a $\boxed{\text{rate}}$ modeled by $\boxed{L(t) = -2t + 15}$, where $L(t)$ is measured in hundreds of bees per hour.

(a) How many bees leave the hive during the time interval $0 \le t \le 2$?

$\int L(t)$

Bees leave $= \int_0^2 L(t)\,dt$ 1pt: integral

$= \int_0^2 (-2t + 15)\,dt$

$= (-t^2 + 15t)\Big|_0^2$ 1pt: antiderivative

$= -2^2 + 15(2) - 0$ ← ok to stop here. 1pt: answer

$= -4 + 30$

$= 26$ hundreds of bees

(b) Write an expression involving one or more integrals for the total number of bees, in hundreds, in the hive at time t for $0 \le t \le 4$. Find the total number of bees in the hive at $t = 4$.

bees in hive = initial + bees enter − bees leave

$B(t) = 35 + \int_0^t E(x)\,dx - \int_0^t L(x)\,dx$ 1pt: expression

Bees in hive @ $t=4$

$B(4) = 35 + \int_0^4 \left(16t - 3t^2 - (-2t + 15)\right) dt$

$= 35 + \int_0^4 (18t - 3t^2 - 15)\,dt$

$= 35 + (9t^2 - t^3 - 15t)\Big|_0^4$ 1pt: antiderivative

$= 35 + 9(4)^2 - 4^3 - 15(4) - 0$ ← ok to stop here 1pt: answer

$= 55$

(c) Find the minimum number of bees in the hive for $0 \le t \le 4$. Justify your answer.

↳ abs min, crit #s & endpts.

$B(t) = 35 + \int_0^t (E(x) - L(x))\,dx$

$B'(t) = E(t) - L(t)$

$0 = 18t - 3t^2 - 15$

$3t^2 - 18t + 15 = 0$

$3(t^2 - 6t + 5) = 0$

$3(t-5)(t-1) = 0$

$t = 5, t = 1$ → crit #s

↳ not in interval $[0,4]$

1pt: sets $E(t) - L(t) = 0$

evaluate endpts.

$B(0) = 35$

$B(4) = 35 + \int_0^4 (E(x) - L(x))\,dx$

$= 55$ from part (b) ☺

evaluate crit #

$B(1) = 35 + \int_0^1 (18t - 3t^2 - 15)\,dt$

$= 35 + (9t^2 - t^3 - 15t)\Big|_0^1$

$= 35 + 9 - 1 - 15 - 0$

$= 28$

1pt: answer

1pt: reason

The minimum # of bees in hive is 28 hundreds of bees.

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