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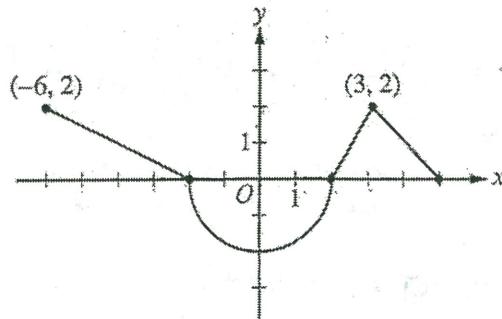
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NO CALCULATOR ALLOWED

Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f' , the derivative of f , consists of a semicircle and three line segments, as shown in the figure above.

- (a) Find the values of $f(-6)$ and $f(5)$.

$$\begin{aligned}
 f(-6) &= f(-2) + \int_{-2}^{-6} f'(t) dt & f(5) &= f(-2) + \int_{-2}^5 f'(t) dt \\
 &= 7 - \int_{-6}^{-2} f'(t) dt & &= 7 + -\frac{1}{2}\pi(2)^2 + \frac{1}{2}(3)(2) \\
 &= 7 - \frac{1}{2}(4)(2) \leftarrow \text{ok to stop here} & &= 7 - 2\pi + 3 \\
 &= 3 & &= 10 - 2\pi
 \end{aligned}$$

(pt → uses initial condition)
 (pt → $f(-6)$)
 (pt → $f(5)$)

- (b) On what intervals is f increasing? Justify your answer.

f inc on $[-6, -2] \cup (2, 5)$ b/c $f' > 0$ on those intervals

2 pts → answer w/
reason

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NO CALCULATOR ALLOWED

- (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

$$f' = 0 \Leftrightarrow x = -2, x = 2 \xrightarrow{\text{rel min + endpts}}$$

$$\begin{array}{c} f' \\ \hline - + - + \end{array}$$

$\hookrightarrow f$ has rel. min @ $x = 2$ b/c f' changes from neg to pos @ $x = 2$

$$f(2) = 7 + \int_{-2}^2 f'(x) dx$$

$$= 7 + -\frac{1}{2}\pi(2)^2$$

$$= 7 - 2\pi$$

from point $P(0)$

$$f(-6) = 3$$

$$f(5) = 10 - 2\pi$$

1 pt - considers $x = 2$

1 pt - answer w/
justification

abs min value is $7 - 2\pi$

Do not write beyond this border:

- (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$$f''(-5) = \frac{2-0}{-6-(-2)} = \frac{-2}{4} = -\frac{1}{2}$$

1 pt $\rightarrow f''(-5)$

$$f''(3) \text{ DNE b/c } \lim_{x \rightarrow 3^-} f''(x) \neq \lim_{x \rightarrow 3^+} f''(x)$$

1 pt $\rightarrow f''(3) \text{ DNE}$
w/
explanation

$$2 \neq -1$$

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NO CALCULATOR ALLOWED

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^{\circ}\text{C}$) at time $t = 0$, and the internal temperature of the potato is greater than 27 $^{\circ}\text{C}$ for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

- (a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

1 pt → tangent line →

$$y - y_1 = m(x - x_1)$$

$$y - 91 = -16(x - 0)$$

$$H - 91 = -16(t - 0)$$

$$\left. \frac{dH}{dt} \right|_{t=0} = -\frac{1}{4}(91 - 27)$$

$$= -\frac{1}{4}(64)$$

$$= -16$$

1 pt → slope

$$H - 91 = -16(3 - 0)$$

1 pt → approximation

$$H = -16(3) + 91 \quad \leftarrow \text{ok to stop here}$$

$$H = -48 + 91$$

$$H(3) = 43^{\circ}\text{C} \quad \leftarrow \text{units optional} \quad \smile$$

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- (b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the

internal temperature of the potato at time $t = 3$,

$$\frac{dH}{dt} = -\frac{1}{4}(H - 27)$$

$$\frac{d^2H}{dt^2} = -\frac{1}{4}\left(\frac{dH}{dt}\right)$$

$$= -\frac{1}{4}\left(-\frac{1}{4}(H - 27)\right)$$

$$= \frac{1}{16}(H - 27)$$

1 pt → underestimate w/reason

$\rightarrow H > 27$ when $t > 0$, $\therefore \frac{d^2H}{dt^2} > 0$ @ $t = 3$

Part (a) is an underestimate of temp of potato @ t = 3 b/c $\frac{d^2H}{dt^2} > 0$ @ t = 3

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NO CALCULATOR ALLOWED

- (c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function

G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius

initial condition and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$? \rightarrow find $G(3)$ \rightarrow find original function \rightarrow antiderivative

$$\frac{dG}{dt} = -(G - 27)^{2/3}$$

$$\int \frac{1}{(G-27)^{2/3}} dG = \int -dt$$

1pt \rightarrow separate variables

$$\int (G-27)^{-2/3} dG = - \int dt$$

$$\int u^{-2/3} du = -t + C$$

$$\begin{aligned} u &= G-27 \\ du &= 1 \\ \frac{du}{dt} &= \frac{dG}{dt} \\ du &= dG \end{aligned}$$

$$3u^{4/3} = -t + C$$

$$3(G-27)^{4/3} = -t + C$$

1pt \rightarrow antiderivative

$$3(91-27)^{4/3} = 0 + C$$

1pt \rightarrow "+C" and initial condition

$$3(64)^{4/3} = C$$

$$3(4) = C$$

$$12 = C$$

$$3(G-27)^{4/3} = -t + 12$$

1pt \rightarrow equation w/ $G + t$

$$(G-27)^{4/3} = -\frac{t+12}{3}$$

$$G-27 = \left(-\frac{t+12}{3}\right)^3$$

$$G = \left(\frac{-t+12}{3}\right)^3 + 27$$

1pt \rightarrow $G(t)$ and $G(3)$

$$G(3) = \left(\frac{-3+12}{3}\right)^3 + 27 = 54^\circ\text{C}$$

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NO CALCULATOR ALLOWED

5. Two particles move along the x -axis. For $0 \leq t \leq 8$, the position of particle P at time t is given by

$$x_P(t) = \ln(t^2 - 2t + 10), \text{ while the velocity of particle } Q \text{ at time } t \text{ is given by } v_Q(t) = t^2 - 8t + 15.$$

Particle Q is at position $x = 5$ at time $t = 0$.

- (a) For $0 \leq t \leq 8$, when is particle P moving to the left?

↳ velocity < 0

$$x_P(t) = \ln(t^2 - 2t + 10)$$

$$x'_P(t) = v_P(t) = (2t - 2) \cdot \frac{1}{t^2 - 2t + 10}$$

$$0 = \frac{2t - 2}{t^2 - 2t + 10}$$

$$2t - 2 = 0$$

$$t = 1$$

$$v_P \begin{array}{c} - \\ \hline \end{array} \begin{array}{c} + \\ \hline \end{array} \begin{array}{c} + \\ \hline \end{array}$$

$$1pt \rightarrow x'_P(t)$$

↳ interval
1pt → interval

Particle P moves left on $[0, 1]$ b/c $v_P(t) > 0$ on $[0, 1)$

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- (b) For $0 \leq t \leq 8$, find all times t during which the two particles travel in the same direction.

$$v_P(t) \begin{array}{c} - \\ \hline \end{array} \begin{array}{c} + \\ \hline \end{array} \begin{array}{c} + \\ \hline \end{array}$$

$$v_Q(t) = t^2 - 8t + 15$$

$$0 = (t - 3)(t - 5)$$

$$t = 3, t = 5$$

$$v_Q \begin{array}{c} + \\ \hline \end{array} \begin{array}{c} - \\ \hline \end{array} \begin{array}{c} + \\ \hline \end{array}$$

↳ $v_P(t) + v_Q(t)$ have
same signs

The two particles travel in same direction

on $(1, 3) \cup (5, 8]$ b/c $v_P(t)$ and $v_Q(t)$ have

the same signs on those intervals

1pt → intervals
1pt → analysis
using
 $v_P(t)$ and
 $v_Q(t)$

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NO CALCULATOR ALLOWED

- (c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$$a_Q(t) = 2t - 8$$

$$\begin{aligned} a_Q(2) &= 2(2) - 8 \\ &= -4 \end{aligned}$$

$$\begin{aligned} v_Q(2) &= 2^2 - 8(2) + 15 \\ &= 3 \end{aligned}$$

$\nearrow a(t) = v'(t)$
 same sign \rightarrow diff signs

1pt $\rightarrow a_Q(2)$

Speed of particle Q is decreasing @ $t = 2$

$$\text{b/c } a_Q(2) < 0 \text{ and } v_Q(2) > 0$$

1pt \rightarrow speed dec w/
reason

- (d) Find the position of particle Q the first time it changes direction.

\downarrow when velocity changes signs

$v_Q(t)$ ^{first} changes signs @ $t = 3$

$$x_Q(3) = x_Q(0) + \int_0^3 v_Q(t) dt$$

$$= 5 + \int_0^3 (t^2 - 8t + 15) dt$$

$$= 5 + \left(\frac{1}{3}t^3 - 4t^2 + 15t\right) \Big|_0^3$$

$$= 5 + \frac{1}{3}(3)^3 - 4(3)^2 + 15(3) - 0$$

$$= 5 + 9 - 36 + 45$$

$$= 23$$

1pt \rightarrow antiderivative
 1pt \rightarrow uses initial condition

ok to
step here 1pt \rightarrow answer

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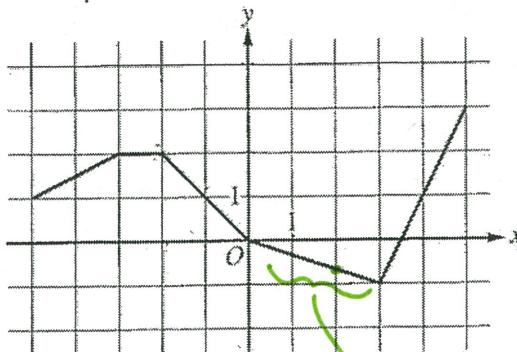
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NO CALCULATOR ALLOWED

x	$g(x)$	$g'(x)$
-5	10	-3
-4	5	-1
-3	2	4
-2	3	1
-1	1	-2
0	0	-3



Graph of h

Slope of h at $x = 2$ = $-\frac{1}{3}$

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x .

Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

- (a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$$f(x) = \cos(2x) + e^{\sin x}$$

$$f'(x) = -2\sin(2x) + \cos x \cdot e^{\sin x}$$

$$f'(\pi) = -2\sin(2\pi) + \cos \pi \cdot e^{\sin \pi} \quad \leftarrow \text{ok to stop here}$$

$$= -e^0$$

$$= -1$$

2pts $\rightarrow f'(\pi)$

chain

- (b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$$k'(x) = f'(x) \cdot h'(f(x))$$

$$k'(\pi) = f'(\pi) \cdot h'(f(\pi))$$

$$= -1 \cdot h'(2)$$

$$= -1 \cdot -\frac{1}{3}$$

$$= \frac{1}{3}$$

1pt $\rightarrow k'(\pi)$

1pt $\rightarrow k'(2)$
 $k'(2)$

$$\begin{aligned} f(\pi) &= \cos 2\pi + e^{\sin \pi} \\ &= 1 + e^0 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

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NO CALCULATOR ALLOWED

- (c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$$m'(x) = h(x) \cdot -2g'(-2x) + g(-2x) \cdot h'(x)$$

2pts $\rightarrow m'(x)$

$$m'(2) = h(2) \cdot -2g'(-2 \cdot 2) + g(-2 \cdot 2) \cdot h'(2)$$

$$= h(2) \cdot -2g'(-4) + g(-4) \cdot h'(2)$$

$$= -\frac{2}{3} \cdot -2(-1) + 5 \cdot -\frac{1}{3}$$

\leftarrow ok to stop
here

$$= -\frac{4}{3} - \frac{5}{3}$$

$$= -\frac{9}{3}$$

$$= -3$$

1pt $\rightarrow m'(2)$

- (d) Is there a number c in the closed interval $[-5, -3]$ such that $g'(c) = -4$? Justify your answer.

$$g'(c) = \frac{g(-5) - g(-3)}{-5 - (-3)}$$

$$1pt \rightarrow \frac{g(-3) - g(-5)}{-3 - (-5)}$$

$$= \frac{10 - 2}{-5 + 3}$$

$$= \frac{8}{-2}$$

$$= -4$$

4pt \rightarrow justification
w/ MVT

g is diff'able, $\therefore g$ is cont

There is a # c on $[-5, -3]$ such that $g'(c) = -4$

by MVT.