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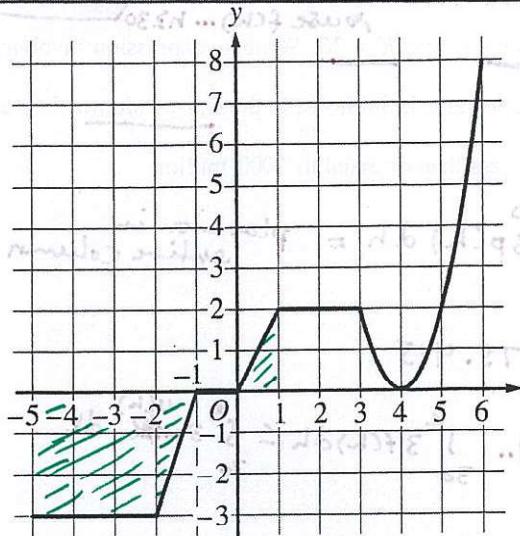
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NO CALCULATOR ALLOWED

Graph of g

3. The graph of the continuous function g , the derivative of the function f , is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \leq x \leq 6$. $\rightarrow g = f'$

- (a) If $f(1) = 3$, what is the value of $f(-5)$?

$$\begin{aligned}
 f(-5) &= f(1) + \int_{-5}^1 f'(x) dx \\
 &= 3 - \int_{-5}^1 g(x) dx \\
 &= 3 - (-3 \cdot 3 + -\frac{1}{2}(1)(3) + \frac{1}{2}(1)(2)) \quad \leftarrow \text{ok to stop here} \\
 &= 3 + 9 + \frac{3}{2} - 1 \\
 &= 11 + \frac{3}{2} = \frac{25}{2}
 \end{aligned}$$

1 pt: integral

1 pt: answer

- (b) Evaluate $\int_1^6 g(x) dx$.

$$\begin{aligned}
 \int_1^6 g(x) dx &= \int_1^3 g(x) dx + \int_3^6 2(x-4)^2 dx \quad u = x-4, \quad du = dx \\
 &= 2 \cdot 2 + \frac{2}{3} (x-4)^3 \Big|_3^6 \\
 &= 4 + \frac{2}{3} ((6-4)^3 - (3-4)^3) \quad \leftarrow \text{ok to stop here} \\
 &= 4 + \frac{2}{3} (8 - 1) \\
 &= 10
 \end{aligned}$$

1 pt: split integral @ $x=3$

1 pt: antiderivative

1 pt: answer

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- (c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

$$\begin{array}{l} f' > 0 \\ g > 0 \end{array} \quad \begin{array}{l} f'' > 0 \\ \text{or } f' \text{ inc} \end{array}$$

and give

f inc and concave up on $(0, 1) \cup (4, 6)$

b/c $f' > 0$ and f' inc on those intervals

1 pt. intervals
1 pt. reason

Sketch at regular $\frac{3}{2}$ min. intervals or sort out to find out

- (d) Find the x -coordinate of each point of inflection of the graph of f . Give a reason for your answer.

$$\begin{array}{l} g = f' \\ g' = f'' \\ f'' = 0 \text{ on } (-1, 0) \text{ and } (1, 3) \\ \text{and } @ x=4 \end{array}$$

$f'' \begin{array}{c} - \\ \bullet \\ + \end{array}$

f has inf pt @ $x=4$ b/c

f'' changes signs @ $x=4$

($\Rightarrow f'$ changes from dec to inc @ $x=4$)

1 pt. answer
1 pt. reason

NO CALCULATOR ALLOWED

| | | | | | |
|--------------------|-----|---|---|----|----|
| t (years) | 2 | 3 | 5 | 7 | 10 |
| $H(t)$ (meters) | 1.5 | 2 | 6 | 11 | 15 |

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$\begin{aligned} H'(6) &= \frac{H(7) - H(5)}{7 - 5} \\ &= \frac{11 - 6}{7 - 5} \quad \leftarrow \text{ok to stop here} \\ &= \frac{5}{2} \text{ meters/year} \end{aligned}$$

$\hookrightarrow H' > 0 \rightarrow H \text{ inc}$
 $H' < 0 \rightarrow H \text{ dec}$

1pt: estimate

The height of the tree is increasing $\frac{5}{2}$ m/year @ $t = 6$ years
 1pt: interpretation w/units

- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.

IVT or MVT

slope... MVT?

H is cont b/c H is twice-diff'able
 H is diff'able b/c H is twice-diff'able

$$H'(t) = \frac{H(5) - H(3)}{5 - 3}$$

$$= \frac{6 - 2}{2}$$

$$= \frac{4}{2}$$

$$= 2$$

1pt: $\frac{H(5) - H(3)}{5 - 3}$

1pt: conclusion w/MVT

\therefore , by MVT, there must be at least one time t ,
 on $2 < t < 10$ s.t. $H'(t) = 2$.

NO CALCULATOR ALLOWED

$$\frac{1}{2}(b_1 + b_2)(w)$$

- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.

$$\frac{1}{2} - a \approx 6$$

$$\text{avg height} = \frac{1}{10-2} \int_2^{10} H(t) dt$$

$$= \frac{1}{8} \left[\frac{1}{2}(1.5+2)(1) + \frac{1}{2}(2+6)(2) + \frac{1}{2}(6+11)(2) + \frac{1}{2}(11+15)(3) \right]$$

1pt: trap sum
1pt: approximation

← ok to stop here

- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where

x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the

base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of

change of the height of the tree with respect to time, in meters per year, at the time when the tree is

derivative
@ 6 = 50
50 meters tall?

$$G(x) = \frac{100x}{1+x}$$

$$\frac{dG}{dx}$$

$$G = 50$$

$$\frac{dx}{dt} = 0.03$$

2pt: $\frac{d}{dt}(G(x))$

$$\frac{dG}{dt} = \frac{(1+x)(100 \cdot \frac{dx}{dt}) - 100x(\frac{dx}{dt})}{(1+x)^2}$$

$$\frac{dG}{dt} \Big|_{x=50} = \frac{(1+1)(100 \cdot 0.03) - 100(1)(.03)}{(1+1)^2}$$

$$= \frac{100(.03)(2-1)}{4}$$

$$= \frac{3}{4}$$

need x ...

$$50 = \frac{100x}{1+x}$$

$$50 + 50x = 100x$$

$$50 = 50x$$

$$1 = x$$

← ok to stop here

1pt: answer

5 5 5 5 5 5 5 5 5

NO CALCULATOR ALLOWED

product

5. Let f be the function defined by $f(x) = e^x \cos x$.

- (a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

$$\begin{aligned} \text{avg rate of change of } f &= \frac{f(\pi) - f(0)}{\pi - 0} \\ &= \frac{e^\pi \cos \pi - e^0 \cos 0}{\pi - 0} && \leftarrow \text{ok to stop here} \\ &= \frac{-e^\pi - 1}{\pi} \end{aligned}$$

1pt: answer

- (b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$\begin{aligned} f'(x) &= \cos x (e^x) + e^x (-\sin x) && \text{1pt: } f'(x) \\ f'\left(\frac{3\pi}{2}\right) &= \cos \frac{3\pi}{2} \cdot e^{\frac{3\pi}{2}} + e^{\frac{3\pi}{2}} (-\sin \frac{3\pi}{2}) && \leftarrow \text{ok to stop here} \quad \text{1pt: } f'\left(\frac{3\pi}{2}\right) \\ &= 0 \cdot e^{\frac{3\pi}{2}} + e^{\frac{3\pi}{2}} (-(-1)) \\ &= e^{\frac{3\pi}{2}} \end{aligned}$$

NO CALCULATOR ALLOWED

- (c) Find the absolute minimum value of f on the interval $0 \leq x \leq 2\pi$. Justify your answer.

Set crit pts & end pts into original, f.

$$f'(x) = 0$$

$$e^x \cos x - e^x \sin x = 0$$

$$e^x (\cos x - \sin x) = 0$$

$$\cancel{e^x = 0} \quad \cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$f(0) = e^0 \cos 0 = 1$$

$$f(2\pi) = e^{2\pi} \cos 2\pi = e^{2\pi}$$

$$f\left(\frac{\pi}{4}\right) = e^{\pi/4} \cos \frac{\pi}{4} = e^{\pi/4} \cdot \frac{\sqrt{2}}{2}$$

$$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4} \cos \frac{5\pi}{4} = e^{5\pi/4} \cdot -\frac{\sqrt{2}}{2}$$

Abs min value of f on $[0, 2\pi]$ is $-\frac{\sqrt{2}}{2} e^{5\pi/4}$.

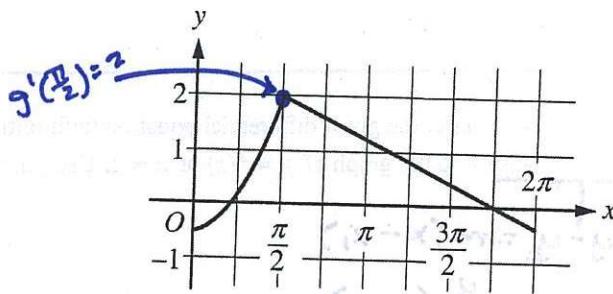
1pt : $f'(x) = 0$

1pt : identifies $x = \frac{\pi}{4}, \frac{5\pi}{4}$

1pt : answer w/
justification

- (d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g' , the derivative of g , is shown

below. Find the value of $\lim_{x \rightarrow \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

$$\lim_{x \rightarrow \pi/2} \frac{f(x)}{g(x)} \rightarrow f\left(\frac{\pi}{2}\right) \rightarrow 0$$

L'Hopital b/c g cont b/c g is diff'ble

$$\lim_{x \rightarrow \pi/2} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \pi/2} \frac{f'(x)}{g'(x)}$$

$$= \frac{f'\left(\frac{\pi}{2}\right)}{g'\left(\frac{\pi}{2}\right)}$$

$$= \frac{e^{\pi/2} \cos \frac{\pi}{2} - e^{\pi/2} \sin \frac{\pi}{2}}{2}$$

$$= -\frac{e^{\pi/2}}{2}$$

1pt : g cont @ $x = \frac{\pi}{2}$
and $\lim g' = 0$

1pt : applies
L'Hopital rule

ok to stop
here

1pt : answer

6

0

0

0

6

0

0

0

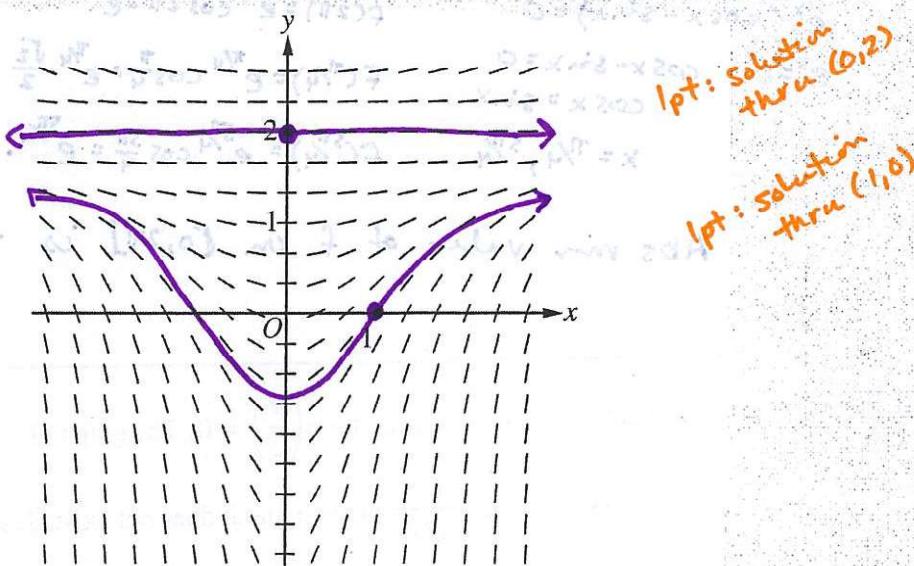
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NO CALCULATOR ALLOWED

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

- (a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



- (b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$.

Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{4}{3}(x - 1)$$

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{(1,0)} &= \frac{1}{3}(1)(0-2)^2 \\ (1,0) &= \frac{4}{3} \end{aligned}$$

$$y = \frac{4}{3}(x-1)$$

1pt: equation of tangent line

$$f(0.7) \approx \frac{4}{3}(0.7-1)$$

ok to stop here

1pt: approximation

$$\approx \frac{4}{3}(-.3)$$

$$\approx -.4$$

NO CALCULATOR ALLOWED

- (c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

\hookrightarrow x's w/ dx, y's w/ dy

$$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$$

$$dy = \frac{1}{3}x(y-2)^2 dx$$

$$\int \frac{1}{(y-2)^2} dy = \int \frac{1}{3}x dx$$

1 pt: separation of variables

$$\int (y-2)^{-2} dy = \frac{1}{3} \int x dx$$

$$\begin{aligned} u &= y-2 \\ du &= dy \end{aligned}$$

$$\int u^{-2} du = \frac{1}{3} \cdot \frac{1}{2} x^2 + C$$

$$-u^{-1} = \frac{1}{6}x^2 + C$$

$$-(y-2)^{-1} = \frac{1}{6}x^2 + C$$

2 pts: antiderivatives
1 pt: "+C" and initial condition

$$-(0-2)^{-1} = \frac{1}{6}(1)^2 + C$$

$$-\frac{1}{2} = \frac{1}{6} + C$$

$$\frac{1}{2} - \frac{1}{6} = C$$

$$\frac{1}{3} = C$$

$$-(y-2)^{-1} = \frac{1}{6}x^2 + \frac{1}{3}$$

$$\frac{1}{y-2} = -\frac{1}{6}x^2 - \frac{1}{3}$$

$$\frac{1}{-\frac{1}{6}x^2 - \frac{1}{3}} = y^{-2}$$

$$\frac{1}{-\frac{1}{6}x^2 - \frac{1}{3}} + 2 = y$$

1 pt: solves for y