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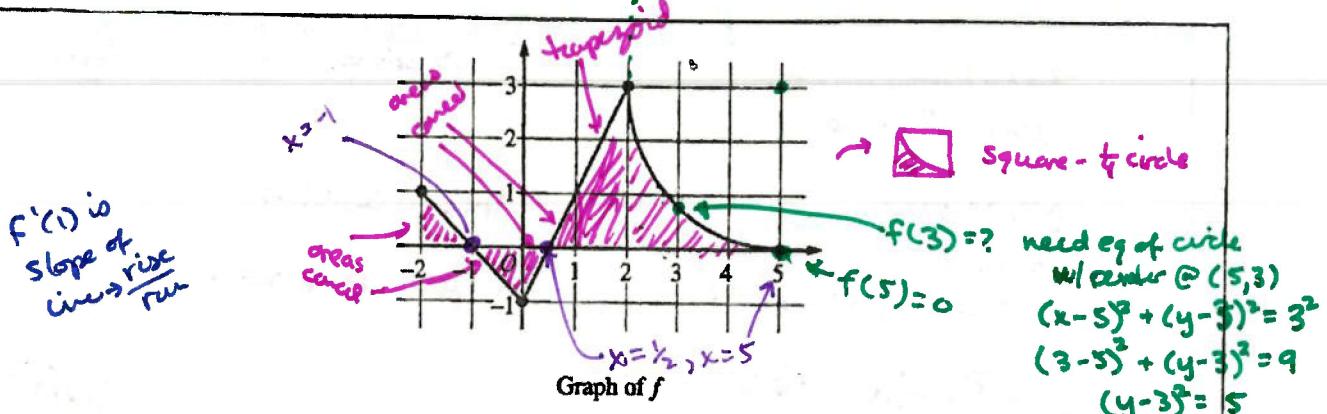
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 $3+\sqrt{5}$

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NO CALCULATOR ALLOWED



3. The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f , consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f .

$$\begin{aligned} y-3 &= \pm\sqrt{5} \\ y &= \sqrt{5} + 3 \text{ or } y = -\sqrt{5} + 3 \\ y &= \end{aligned}$$

- (a) If $\int_{-6}^5 f(x) dx = 7$, find the value of $\int_{-6}^{-2} f(x) dx$. Show the work that leads to your answer.

$$\begin{array}{c} -6 \xrightarrow{?} -2 \xrightarrow{\text{from graph}} 5 \\ \hline \end{array}$$

$$\begin{aligned} \int_{-6}^{-2} f(x) dx &= \int_{-6}^5 f(x) dx - \int_{-2}^5 f(x) dx && \leftarrow 1\text{pt:} \\ &= 7 - \left[\frac{1}{2}(1+3)(1) + 9 - \frac{1}{4}\pi(3)^2 \right] && \leftarrow \text{ok to stop here} \\ &= 7 - [2 + 9 - \frac{9\pi}{4}] && \leftarrow 1\text{pt: } \int_{-2}^5 f(x) dx \\ &= 7 - 11 + \frac{9\pi}{4} && \leftarrow 1\text{pt: answer} \\ &= -4 + \frac{9\pi}{4} \end{aligned}$$

- (b) Evaluate $\int_3^5 (2f'(x) + 4) dx$.

$$\begin{aligned} \int_3^5 (2f'(x) + 4) dx &= 2 \int_3^5 f'(x) dx + \int_3^5 4 dx \\ &= 2f(x)|_3^5 + 4x|_3^5 && \leftarrow 1\text{pt: FTC} \\ &= 2[f(5) - f(3)] + 4(5) - 4(3) \\ &= 2[0 - (3 - \sqrt{5})] + 4(5) - 4(3) && \leftarrow \text{ok to stop here} \\ &= -6 + 2\sqrt{5} + 20 - 12 && \leftarrow 1\text{pt: answer} \\ &= 2 + 2\sqrt{5} \end{aligned}$$

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NO CALCULATOR ALLOWED

- (c) The function g is given by $g(x) = \int_{-2}^x f(t) dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

→ crit #s + endpts in original equation

$$g(x) = \int_{-2}^x f(t) dt$$

$$g'(x) = f(x)$$

$$f(x) = 0$$

$$\text{at } x = -1, x = \frac{1}{2}, x = 5$$

$$1\text{pt: } g'(x) = f(x)$$

$$1\text{pt: crit # } x = -1$$

$$g(-1) = \int_{-2}^{-1} f(t) dt = \cancel{-}\frac{1}{2}(1)(1) = \frac{1}{2}$$

$$g(\frac{1}{2}) = \int_{-2}^{\frac{1}{2}} f(t) dt = -\frac{1}{2}(\frac{1}{2})(1) = -\frac{1}{4}$$

did this
in part a.

$$g(5) = \frac{1}{2}(1+3)(1) + 9 - \frac{1}{4}\pi(3)^2 = 11 - \frac{9\pi}{4}$$

← abs max value
of g is $11 - \frac{9\pi}{4}$

$$g(-2) = 0$$

1pt: answer w/ justification

- (d) Find $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

slope @ $x=1$ ☺

$$\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$$

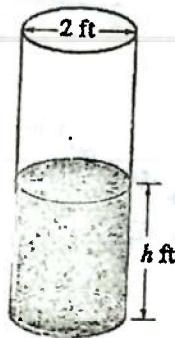
$$= \frac{10 - 3(2)}{1 - \arctan 1} \quad \leftarrow \begin{matrix} \text{ok to} \\ \text{stop here} \end{matrix}$$

$$= \frac{4}{1 - \frac{\pi}{4}}$$

$$= \frac{16}{4 - \pi}$$

1pt: answer

NO CALCULATOR ALLOWED



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

- (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$$h = 4$$

$$r = 1 \quad \begin{matrix} \nearrow \\ r \text{ is constant} \end{matrix}$$

or a cylinder

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = ?$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$$

$$1 \text{ pt: } \frac{dV}{dt}$$

1 pt: answer w/
units

$$\begin{aligned} & \frac{dV}{dt} = \pi(1^2)(-\frac{1}{10}\sqrt{4}) \leftarrow \text{ok to stop here} \\ & = -\frac{\pi}{5} \frac{\text{ft}^3}{\text{sec}} \end{aligned}$$

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NO CALCULATOR ALLOWED

- (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

rate inc $\rightarrow (\text{rate})' > 0$
 rate dec $\rightarrow (\text{rate})' < 0$

} derivative of rate is 2nd derivative

$$\frac{d^2h}{dt^2} = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}} \frac{dh}{dt}$$

$$\left. \frac{d^2h}{dt^2} \right|_{h=3} = -\frac{1}{10} \cdot \frac{1}{2} (3)^{-\frac{1}{2}} \left(-\frac{1}{10}\sqrt{4} \right)$$

$$> 0$$

2 pts. $\frac{d^2h}{dt^2}$

Rate of change of height of water is inc when $h=3$

b/c $\left. \frac{d^2h}{dt^2} \right|_{h=3} > 0$.

1 pt: answer w/
reason

- (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .

initial condition

$\hookrightarrow x w/dx, y w/dy$

$$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$$

1 pt: separate variables

$$\frac{1}{\sqrt{h}} dh = -\frac{1}{10} dt$$

1 pt: antiderivatives

$$\int h^{-\frac{1}{2}} dh = \int -\frac{1}{10} dt$$

1 pt: "+C" and initial condition

$$2h^{\frac{1}{2}} = -\frac{1}{10}t + C \rightarrow 2h^{\frac{1}{2}} = -\frac{1}{10}t + 2\sqrt{5}$$

1 pt: solves for h

$$2(5)^{\frac{1}{2}} = -\frac{1}{10}(0) + C$$

$$h^{\frac{1}{2}} = \frac{-\frac{1}{10}t + 2\sqrt{5}}{2}$$

$$2\sqrt{5} = C$$

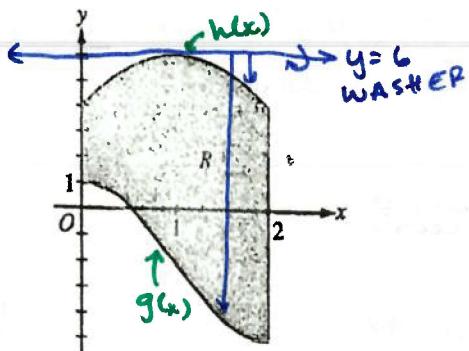
$$h = \left(\frac{-\frac{1}{10}t + 2\sqrt{5}}{2} \right)^2$$

or

$$h = \left(-\frac{1}{20}t + \sqrt{5} \right)^2$$

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NO CALCULATOR ALLOWED



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3 \cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y -axis, and the vertical line $x = 2$, as shown in the figure above.

- (a) Find the area of R .

$$\begin{aligned}
 \text{Area of } R &= \int_0^2 [6 - 2(x-1)^2 - (-2 + 3 \cos(\frac{\pi}{2}x))] dx && \text{1 pt: integrand} \\
 &= \int_0^2 [6 - 2(x-1)^2 + 2 - 3 \cos(\frac{\pi}{2}x)] dx && \text{1 pt: antiderivative cos} \\
 &= \int_0^2 [8 - 2(x-1)^2 - 3 \cos(\frac{\pi}{2}x)] dx && \text{1 pt: antiderivative other terms} \\
 &= 8x \Big|_0^2 - 2 \int_0^2 (x-1)^2 dx - 3 \int_0^2 \cos(\frac{\pi}{2}x) dx \\
 &\quad \stackrel{u=x-1}{\substack{u=x-1 \\ du=dx}} \quad \stackrel{u=\frac{\pi}{2}x}{\substack{du=\frac{\pi}{2}dx \\ \frac{2}{\pi}du=dx}} \\
 &= 16 - 2 \int_{-1}^1 u^2 du - 3 \cdot \frac{2}{\pi} \int_0^{\pi} \cos u du \\
 &\quad \stackrel{u^2}{=} \stackrel{u}{=} \stackrel{2}{=} \\
 &= 16 - 2 \cdot \frac{1}{3}u^3 \Big|_{-1}^1 - \frac{6}{\pi} \sin u \Big|_0^{\pi} && \text{1 pt: answer} \\
 &= 16 - \frac{2}{3}((+1)^3 - (-1)^3) - \frac{6}{\pi}(\sin \pi - \sin 0) && \leftarrow \text{ok to stop here} \\
 &= 16 - \frac{2}{3}(1+1) - \frac{6}{\pi}(0) \\
 &= 16 - \frac{4}{3}
 \end{aligned}$$

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NO CALCULATOR ALLOWED

- (b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x -axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$\text{Volume} = \int \text{area cross section}$$

$$\text{Volume} = \int_0^2 A(x) dx$$

$$= \int_0^2 \frac{1}{x+3} dx \quad u = x+3 \\ du = dx$$

$$= \int_3^5 \frac{1}{u} du$$

$$= \ln|u| \Big|_3^5$$

$$= \ln 5 - \ln 3 \quad \text{or } \ln(5/3)$$

1pt: integral

1pt: answer

- (c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$\text{Volume} = \pi \int_0^2 [(6-g(x))^2 - (6-h(x))^2] dx$$

do not
evaluate!

1pt: limits π

1pt: form of integrand

1pt: integrand

6



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6



6



6



NO CALCULATOR ALLOWED

6. Functions f , g , and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.
- To slope of tangent line

- (a) Find $h'(2)$.

\hookrightarrow slope of tangent line to h at $x = 2$

$$h'(2) = \frac{2}{3}$$

= lpt: answer

- (b) Let a be the function given by $a(x) = \underbrace{3x^3}_{\text{product}} h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$$a'(x) = h(x) \cdot 9x^2 + 3x^3 \cdot h'(x)$$

lpt: form of product rule

$$a'(2) = h(2) \cdot 9(2)^2 + 3(2)^3 \cdot h'(2)$$

lpt: $a'(x)$

$$= 4 \cdot 9(2)^2 + 3(2)^3 \cdot \frac{2}{3} \quad \leftarrow \begin{matrix} \text{ok to stop} \\ \text{here} \end{matrix}$$

lpt: $a'(2)$

$$= 36 \cdot 4 + 3 \cdot 8 \cdot \frac{2}{3}$$

$$= 144 + 16$$

$$= 160$$

NO CALCULATOR ALLOWED

- (c) The function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \rightarrow 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \rightarrow 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$$\lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{1 - (f(x))^3} \right) \rightarrow \frac{0}{0}$$

$$\therefore \lim_{x \rightarrow 2} (x^2 - 4) = 0 \quad \text{and} \quad \lim_{x \rightarrow 2} (1 - (f(x))^3) = 0$$

$$\Rightarrow \lim_{x \rightarrow 2} f(x)^3 = 0$$

$$1 - (\lim_{x \rightarrow 2} f(x))^3 = 0$$

$$1 = (\lim_{x \rightarrow 2} f(x))^3$$

$$1 = \lim_{x \rightarrow 2} f(x)$$

$$f \text{ is cont b/c } f \text{ diff'able}$$

$$\therefore f(2) = 1 = \lim_{x \rightarrow 2} f(x)$$

f' cont b/c

f twice-differentiable

$$\therefore \lim_{x \rightarrow 2} f'(x) = f'(2)$$

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{1 - (f(x))^3} &= \lim_{x \rightarrow 2} \frac{2x}{3 - f'(x) \cdot 3(f(x))^2} \\ &= \frac{4}{3 - f'(2) \cdot 3(f(2))^2} \\ &= \frac{4}{3 - 3f'(2)} \end{aligned}$$

$$\therefore f(2) = 1$$

Since h is cont b/c h is diff'able,

$$\lim_{x \rightarrow 2} h(x) = h(2)$$

$$\frac{4}{-3f'(2)} = 4$$

$$-3f'(2) = 4$$

$$-\frac{1}{3} = f'(2)$$

- (d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

g and h are cont b/c g and h are diff'able

$$\therefore \lim_{x \rightarrow 2} g(x) = g(2) = 4$$

$$\text{and } \lim_{x \rightarrow 2} h(x) = h(2) = 4$$

$$g(x) \leq k(x) \leq h(x)$$

$$g(2) \leq k(2) \leq h(2)$$

$$\therefore k(2) = 4$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow 2} g(x) &\leq \lim_{x \rightarrow 2} k(x) \leq \lim_{x \rightarrow 2} h(x) \\ \therefore \lim_{x \rightarrow 2} k(x) &= 4 \end{aligned}$$

$$\lim_{x \rightarrow 2} k(x) = k(2) \quad \text{so } k \text{ is cont @ } x = 2$$

lpt: continuous w/
reason