U and US in Calculus

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Rubrics for Sample Student Work

Question 5

Score	Qualifications
0	No work or no anti-differentiation or u-substitution not used
1	Error in choosing u or du/dx. No u #'s found.
2	Error in substitution or simplifying.
3	Error in FTC OR computation error.
4	Error in notation.
5	Proper notation used throughout problem.

Question 2

Score	Qualifications
0	No work or understanding about both the sign of $v(t)$ and $a(t)$ (referencing only $v(t)$ or $a(t)$)
1	Recognizes need to use $a(t)$ and $v(t)$, but writes wrong information about $a(t)$ and $v(t)$ at correct t-value
2	Recognizes need to use $a(t)$ and $v(t)$, writes correct information about $a(t)$ or $v(t)$, but writes incorrect conclusion OR no t-value referenced at all OR only says $a(t)$ & $v(t)$ have opposite/same signs.
3	Writes correct information about $a(t)$ or $v(t)$ and correct conclusion.
4	Error in notation.
5	Proper notation used throughout problem.

Question 3

Score	Qualifications
0	No work or inaccurate separation of variables
1	Error in antiderivative OR missing u-substitution OR missing + C
2	Does not solve for C (does not use initial condition) OR error in u-substitution.
3	Error in solving for C OR error in solving for P.
4	Error in notation.
5	Proper notation.

5

5. Evaluate the integral: $\int_0^1 \frac{x^2}{2-x^3}\,dx$

$u = 2-x^3$ $\quad u(1) = 1$, $u(0) = 2$

$\frac{du}{dx} = -3x^2$

$\frac{du}{-3x^2} = dx$

$\int_0^1 x^2(2-x^3)^{-1}\,dx$

$= \int_2^1 x^2 u^{-1} \frac{du}{-3x^2}$

$= -\frac{1}{3}\int_2^1 u^{-1}\,du$

$= -\frac{1}{3} \ln|u| \Big|_2^1$

$= -\frac{1}{3}(\ln(1) - \ln(2)) = -\frac{1}{3}(0-\ln 2) = \frac{1}{3}\ln 2$

$= -\frac{1}{3}\ln\left(\frac{1}{2}\right)$

$\boxed{= -\frac{\ln\left(\frac{1}{2}\right)}{3}}$ ok, but $\ln 1 = 0$

5

5. Evaluate the integral: $\int_0^1 \frac{x^2}{2-x^3}\,dx$

$u = 2-x^3$ $\quad u(1) = 2-(1)^3 = 1$, $u(0) = 2-(0)^3 = 2$

$\frac{du}{dx} = -3x^2$

$\frac{du}{-3x^2} = dx$

$= \int_2^1 \frac{\cancel{x^2}}{u} \cdot \frac{du}{-3\cancel{x^2}}$

$= -\frac{1}{3}\int_2^1 \frac{1}{u}\,du$

$= -\frac{1}{3}(\ln|u|)\Big|_2^1$

$= -\frac{1}{3}[\ln|1| - \ln|2|]$

$= -\frac{1}{3}[0-\ln 2]$

$= -\frac{1}{3}[-\ln 2]$ $\boxed{= \frac{\ln 2}{3}}$

3

5. Evaluate the integral: $\int_0^1 \frac{x^2}{2-x^3}\,dx$

$= \int_0^1 x^2(2-x^3)^{-1}\,dx$

$u = 2-x^3$

$\frac{du}{dx} = -3x^2$

$\frac{du}{-3x^2} = dx$

$u(1) = 2-(1)^3 = 2-1 = 1$

$u(0) = 2-(0)^3 = 2-0 = 2$

$= \int_2^1 \cancel{x^2}\, u^{-1} \frac{du}{-3\cancel{x^2}}$

$= \int_2^1 u^{-1} \frac{du}{-3}$

$= -\frac{1}{3}\int_2^1 \frac{1}{u}\,du$

$= -\frac{1}{3}(\ln u)\Big|_2^1$

$= -\frac{1}{3}(\ln|2-x^3|)\Big|_2^1$

why did your x come back?

$= -\frac{1}{3}(\ln|2-2^3| - (\ln|2-1^3|))$

$= -\frac{1}{3}(\ln|-6| - \ln 1)$

$= -\frac{1}{3}(\ln 6 - \ln 1)$

$= -\frac{1}{3}(\ln \frac{6}{1})$

$= -\frac{1}{3}(\ln 6)$ or $= -\frac{\ln 6}{3}$

3

5. Evaluate the integral: $\int_{-1}^0 \frac{x^2}{1-x^3}\,dx$

$\int_{-1}^{0} \frac{x^2}{1-x^3}\,dx$

$= \int_{-1}^{0} x^2(1-x^3)^{-1}\,dx$

$u = 1-x^3$

$\frac{du}{dx} = -3x^2$

$dx = \frac{du}{-3x^2}$

$u(0) = 1$

$u(-1) = 2$

$= \int_2^1 \cancel{x^2}(u)^{-1} \frac{du}{-3\cancel{x^2}}$

$= \int_2^1 u^{-1} \frac{du}{-3}$

$= -\frac{1}{3}\int_2^1 u^{-1}\,du$

$= -\frac{1}{3}\int_2^1 \frac{1}{u}\,du$

$= -\frac{1}{3}(\ln|u|)\Big|_2^1$

$= -\frac{1}{3}(\ln|x|)\Big|_2^1$

$= -\frac{1}{3}(\ln|1| - \ln|2|)$

$= -\frac{1}{3}(\ln|1| - \ln|2|)$

$= \boxed{-\frac{1}{3}\ln|1| + \frac{1}{3}\ln|2|}$

Simplify

2 5. Evaluate the integral: $\int_{-1}^{0} \frac{x^3}{1+x^4}\,dx$

$\int_{-1}^{0} \frac{x^3}{1+x^4}\,dx$ $\quad u = 1+x^4 \quad u(0)=1$

$\frac{du}{dx} = 4x^3 \quad u(-1)=2$

$\int_{1}^{2} \frac{\cancel{x^3}}{u} \frac{du}{4\cancel{x^3}}$ $\quad \frac{du}{4x^3} = dx$

$= \frac{1}{4}\int_{1}^{2} \frac{1}{u}\,du$

$= \frac{1}{4}(\ln|u|)\Big|_1^2$

$= \frac{1}{4}(\ln|2| - \ln|1|)$

$= \frac{1}{4}(\ln|\frac{2}{1}|)$

$= \frac{1}{4}(\ln|2|)$

$= \boxed{\frac{\ln|2|}{4}}$

---

2 5. Evaluate the integral: $\int_{0}^{1} \frac{x^2}{2-x^3}\,dx$

$u = 2-x^3$

$\int_0^1 \frac{x^2}{2-x^3}\,dx = \int_0^1 x^2(2-x^3)^{-1}\,dx$

$\frac{du}{dx} = -3x^2$

$= \int_2^1 \cancel{x^2}(u)^{-1} \frac{du}{-3\cancel{x^2}}$

$\frac{du}{-3x^2} = dx$

$= -\frac{1}{3}\int_2^1 (u)^{-1}\,du$

you never antiderived

$u = 2-(1)^3$
$= 2-1$
$= 1$

$= -\frac{1}{3} \cdot \frac{\cancel{1}}{u}\Big|_2$

$= -\frac{1}{3}\left(\frac{1}{2-(2)^3} - \frac{1}{2-(1)^3}\right) = -\frac{1}{3}\left(\frac{1}{-6} - \frac{1}{3}\right)$

$= -\frac{1}{2}\left(\frac{2}{6}\right) = \boxed{-\frac{2}{18}}$

$u = 2-(0)^3$
$= 2-0$
$= 2$

5. Evaluate the integral: $\int_0^1 \frac{x^2}{2-x^3}\,dx$

$$\int_0^1 x^2(2-x^3)^{-1}\,dx$$

$$U = 2-x^3$$

$$\frac{du}{dx} = -3x^2 \rightarrow \frac{du}{-3x^2} = dx$$

$$= \int_0^1 \cancel{x^2} \cdot U^{-1} \cdot \frac{du}{-3\cancel{x^2}}$$

need u #s

$$= -\frac{1}{3}\int_0^1 \frac{1}{U}\,du$$

$$= -\frac{1}{3}\ln|U|\Big|$$

$$= -\frac{1}{3}\ln(2-x^3)\Big|_0^1$$

definite integral... never go back to your x's.

$$= -\frac{1}{3}\ln(2-(1)^3) - \left(-\frac{1}{3}\ln(2-0^3)\right)$$

$$= \boxed{-\frac{1}{3}\ln 1 + \frac{1}{3}\ln 2}$$

---

5. Evaluate the integral: $\int_{-1}^0 \frac{x^3}{1+x^4}\,dx$

$u = 1+x^4$ $\quad U = \cancel{x^3}$ $\quad U(0) = 0$

$$\frac{du}{dx} = 3x^4 \qquad U(1) = -1$$

$$\frac{du}{3x^4} = dx$$

$$= \int_{-1}^0 \frac{U}{1+x^4}\,\frac{du}{3x^4}$$

$$= \frac{1}{3}\int_{-1}^0 \frac{U}{1+x^4}\,\frac{du}{x^4}$$

$$=$$

$$U = 1+x^4$$

$$\frac{du}{dx} =$$

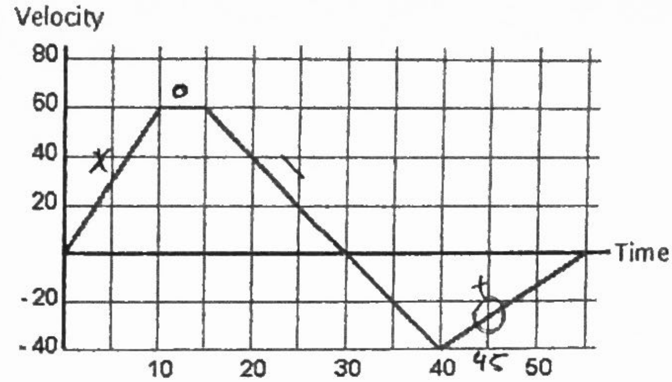
[redacted]

1

2. The figure shows the velocity, in meters per second, of a particle moving along a coordinate line. At $t = 45$, is the particle speeding up, slowing down, or neither?

(+) a(t) & v(t) same sign

a(t) & v(t) opp. signs



velocity <0 and ~~acceleration~~ acceleration >0 are ~~increasing~~ on ~~(40, 50)~~ at $t=45$

∴ at $t = 45$ the particle is speeding up b/c $v(t)$ & $a(t)$ have the same sign

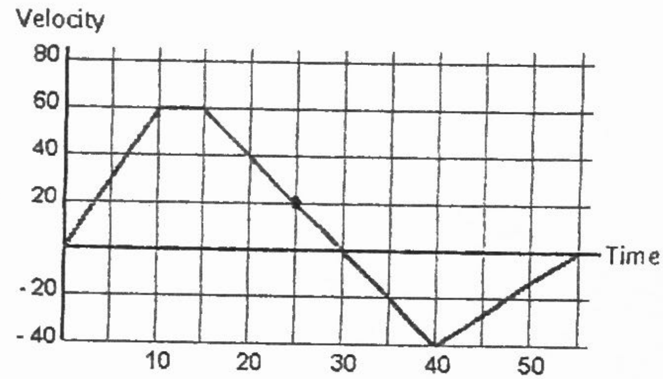
2

2. The figure shows the velocity, in meters per second, of a particle moving along a coordinate line. At $t = 25$, is the particle speeding up, slowing down, or neither?

what is?

when? all the time?

Slowing down b/c $v(t)>0$ and $a(t)<0$.

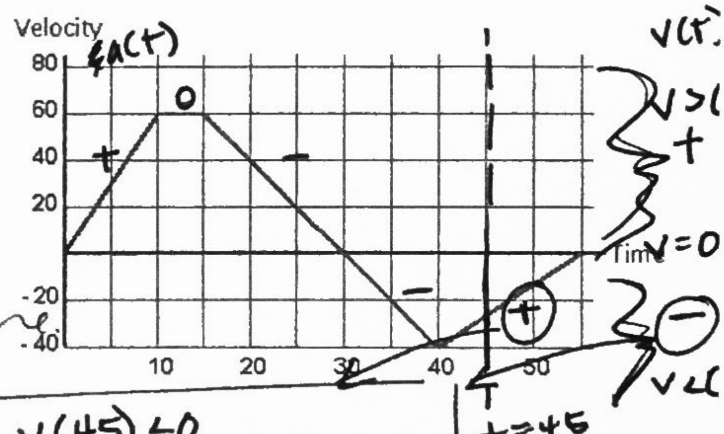


[redacted]

2

2. The figure shows the velocity, in meters per second, of a particle moving along a coordinate line. At $t = 45$, is the particle speeding up, slowing down, or neither?

@ $t=45$ the particle is slowing down because the sign of acceleration >0 & velocity <0 are not the same at $t=45$



$v(45)<0$
$a(45)>0$
∴ slowing down b/c $v(t)$ & $a(t)$ have different signs @ $t=45$

4 3. A function $P(t)$ models the total amount of paper, where P is measured in tons and t is measured in months, used at a local school. School administrators estimate that P will satisfy the differential equation $\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$. Find a solution $P(t)$ to the differential equation satisfying $P(0) = \frac{3}{2}$.

t

t P

$u = 2p$

$\frac{du}{2} = dP$

$\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$

$\int e^{2P}\,dP = \int 3t^2\,dt$

$\int e^{u}\frac{du}{2} = t^3 + C$

$\frac{1}{2}\int e^{u}\,du = t^3 + C$

$\frac{1}{2}[e^{u}] = t^3 + C$

$\frac{1}{2}[e^{2P}] = t^3 + C$

$\frac{1}{2}e^{2(3/2)} = 0^3 + C$

$\frac{e^{2(3/2)}}{2} = C$

$\frac{1}{2}[e^{2P}] = t^3 + \frac{e^3}{2}$

$e^{2P} = \left[t^3 + \frac{e^3}{2}\right]2$

$\ln(e^{2P}) = \ln[2t^3 + e^3]$

$\frac{2P}{2} = \frac{\ln[2t^3 + e^3]}{2}$

$P(t) = \frac{\ln[2t^3 + e^3]}{2}$

3 3. A function $P(t)$ models the total amount of paper, where P is measured in tons and t is measured in months, used at a local school. School administrators estimate that P will satisfy the differential equation $\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$. Find a solution $P(t)$ to the differential equation satisfying $P(0) = \frac{1}{2}$

$(0, \frac{1}{2})$

$$dt \cdot \frac{dP}{dt} = \frac{3t^2}{e^{2P}} \cdot dt$$

$$e^{2P} \cdot dP = \frac{3t^2}{e^{2P}} dt \cdot e^{2P}$$

$$e^{2P} dP = 3t^2 dt$$

$2P = u$

$2 = \frac{du}{dP}$

$dP = \frac{du}{2}$

$$\int e^{2P} dP = \int 3t^2 dt$$

$$\int e^{u} \frac{du}{2} = \int 3t^2 dt$$

$$\frac{1}{2} \int e^{u} du = \int 3t^2 dt$$

$$\frac{1}{2} \cdot e^{u} = 3 \cdot \frac{1}{1+2} t^{3} + C$$

$$\frac{1}{2} e^{u} = t^3 + C$$

$$\frac{1}{2} e^{2P} = t^3 + C$$

$$\frac{1}{2} e^{2(1/2)} = 0^3 + C$$

$$\frac{1}{2} e^{1} = C$$

$$2 \cdot \frac{1}{2} e^{2P} = (t^3 + \frac{1}{2} e^{1}) 2$$

$$e^{2P} = (2t^3 + e^{1})$$

ln a side, not each term

$\ln \qquad \ln$

$$2P = \cancel{\ln 2t^3 + \ln e^1} \ln|2t^3 + e|$$

$$2P = \ln^{2t^3} + 1$$

$$P(t) = \frac{\ln^{2t^3} + 1}{2}$$

2 3. A function $P(t)$ models the total amount of paper, where P is measured in tons and t is measured in months, used at a local school. School administrators estimate that P will satisfy the differential equation $\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$. Find a solution $P(t)$ to the differential equation satisfying $P(0) = \frac{1}{2}$

$t=0 \quad P=\frac{1}{2}$

$$\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$$

$u=2P$

$\frac{du}{dt}=2$

$\frac{du}{2}=dt$

$$dp = \frac{3t^2}{e^{2P}} \cdot dt$$

$$e^{2P} \cdot dp = 3t^2\, dt$$

$$\int e^u \cdot \cancel{dP}\frac{du}{2} = \int 3t^2\, du$$

$$\frac{1}{2}e^u = \frac{1}{2}\int 3t^2 du$$

$$\frac{1}{2}e^u = \frac{1}{2}\cdot t^3 + C$$

$$\frac{1}{2}e^{2P} = \frac{1}{2}t^3 + C$$

$$\frac{1}{2}e^{2(\frac{1}{2})} = \frac{1}{2}(0)^3 + C$$

$$\frac{1}{2}e^1 = C$$

$$\frac{1}{2}(2.718) = C$$

$$e^{2P} = \frac{1}{2}t^3 + 2\cancel{.718}$$

√? how did e turn into ln?

$$\ln|2P| = \ln_e|\frac{1}{2}t^3 + \cancel{2.718}|$$

$$|2P| = |\frac{1}{2}t^3 + 2\cancel{.718}|$$

$$\frac{2P}{2} = \frac{\frac{1}{2}t^3 + 2\cancel{.718}}{2}$$

$$\boxed{P = \frac{t^3 + 2.718}{2}}$$

3. A function $P(t)$ models the total amount of paper, where P is measured in tons and t is measured in months, used at a local school. School administrators estimate that P will satisfy the differential equation $\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$. Find a solution $P(t)$ to the differential equation satisfying $P(0) = \frac{1}{2}$. $(0, \frac{1}{2})$

$t \quad P$

$\frac{dP}{dt} \cdot dt = \frac{3t^2}{e^{2P}} \cdot dt$

$e^{2P} \cdot dP = \frac{3t^2}{e^{2P}} dt \cdot e^{2P}$

$u = 2P$

$\frac{du}{dP} = 2$

$\frac{du}{2} = dP$

$\int e^{2P} dP = \int 3t^2 dt$

$\int e^u \frac{du}{2} = 3\int t^2 dt$

$\frac{1}{2} \int e^u du = 3(\frac{1}{3} t^3) + C$

$\frac{1}{2} \cancel{\ln|u|} = t^3 + C$

$\frac{1}{2} \ln|2P| = t^3 + C$ $\qquad$ $\frac{1}{2} \ln|2(\frac{1}{2})| = 0^3 + C$

$\cancel{2} \cdot \frac{1}{2} \ln|2P| = (t^3 + 0) 2$ $\qquad$ $\frac{1}{2} \ln|1| = C$

$\qquad$ $\frac{1}{2}(0) = C$

$e^{\ln|2P|} = e^{2t^3}$ $\qquad$ $0 = C$

$|2P| = e^{2t^3}$

$2P = \pm e^{2t^3}$ $\qquad$ $(0 + \frac{1}{2} = \frac{1}{2} > 0)$

$\frac{2P}{2} = \frac{e^{2t^3}}{2}$

$P = \frac{e^{2t^3}}{2}$

1

3. A function $P(t)$ models the total amount of paper, where P is measured in tons and t is measured in months, used at a local school. School administrators estimate that P will satisfy the differential equation $\frac{dP}{dt} = \frac{3t^2}{e^{2P}}$. Find a solution $P(t)$ to the differential equation satisfying $P(0) = \frac{1}{2}$.

$$dt \cdot \frac{dP}{dt} = \frac{3t^2}{e^{2P}} \cdot dt$$

$$e^{2P} \cdot dP = \frac{3t^2}{e^{2P}} \cdot dt \cdot e^{2P}$$

$$\int e^{2P}\,dP = \int (3t^2)\,dt$$

$u = 2P$

$du = 2\,dP$

$\frac{du}{2} = dP$

$$\int (e^{u}) \frac{du}{2} = t^3 + C$$

$$\int e^{(2P)} \frac{du}{2} = t^3$$

$$e^{P} = t^2$$

$$P(0) = e^{(1/2)} = t - (0)^2 \cdot 0$$

$$P(0) = 1.649$$

Create A Rubric

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$. Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

Score	**Qualifications**
0	
1	
2	
3	
4	
5	

Create A Rubric

The position equation for the movement of a particle is given by $x(t) = \frac{\cos t}{1-\sin t}$ where x is measured in feet and t is measure in seconds. Find the velocity equation.

Score	**Qualifications**
0	
1	
2	
3	
4	
5	

Create A Rubric
(Sample Rubrics)

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$. Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

Solution: $Area = \int_{1/2}^{1} (e^x - \ln x)\, dx$

$= 1.223$

Score	Qualifications
0	No work or no understanding of area btn curves (i.e., missing limits, no integral)
1	Correct limits, but wrong function (i.e., wrong order of functions, $g(x) - f(x)$ OR missing one of the functions) OR correct function, but error in limits
2	Correct limits & order of functions, $f(x) - g(x)$, but does not calculate an answer
3	Computation error (AP-style)
4	Error in notation.
5	Proper notation

The position equation for the movement of a particle is given by $x(t) = \frac{\cos t}{1 - \sin t}$ where x is measured in feet and t is measured in seconds. Find the velocity equation.

Solution: $v(t) = x'(t)$

$$= \frac{(1 - \sin t)(-\sin t) - \cos t\,(-\cos t)}{(1 - \sin t)^2}$$

$$= \frac{-\sin t + \sin^2 t + \cos^2 t}{(1 - \sin t)^2}$$

$$= \frac{-\sin t + 1}{(1 - \sin t)^2}$$

$$= \frac{1}{1 - \sin t}$$

Score	Qualifications
0	Not doing quotient rule
1	Error in quotient rule OR error in one of the derivatives (f' or g')
2	Error in simplification (i.e. doesn't distribute "-" correctly)
3	Error in trig simplification. (i.e. doesn't replace $\sin^2 x + \cos^2 x$ with 1)
4	Error in notation.
5	Proper notation used throughout problem.