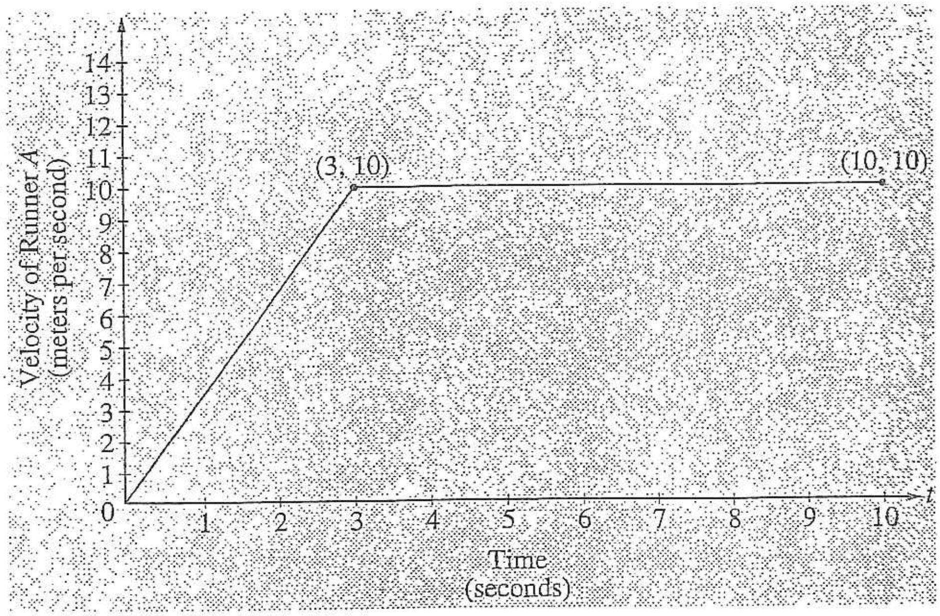
Topic 1: Particle Motion
Particle moving on a line

These questions may give the position equation, the velocity equation or the acceleration equation along with an initial condition. Students may be asked about the motion of the particle: its direction, when it changes direction, its maximum position in one direction (farthest left or right) etc. Speed, the absolute value of velocity, is also a common topic.

The particle may be a "particle," a person, car, etc. The position, velocity or acceleration may be given as an equation, a graph or a table. There are a lot of different things students may be asked to find. While particles don't really move in this way, the question is a versatile way to test a variety of calculus concepts.

What students should know how to do (AB and BC):

- Initial value differential equation problems: given the velocity or acceleration with initial condition(s) find the position or velocity.
- Distinguish between position at some time (displacement) and the total distance traveled during the time.
  - The total distance traveled is the definite integral of the absolute value of the rate of change (velocity): $\int_a^b |v(t)|\,dt$
  - The net distance (displacement) is the definite integral of the rate of change (velocity): $\int_a^b v(t)\,dt$
  - The final position is the initial position plus the definite integral of the rate of change from $x = a$ to $x = t$: $s(t) = s(a) + \int_a^t v(x)\,dx$ Notice that this is an accumulation function equation.
- Find the speed at a particular time. The speed is the absolute value of the velocity.
- Find average speed, velocity, or acceleration
- Determine if the speed is increasing or decreasing.
  - If at some time, the velocity and acceleration have the *same* sign then the speed is increasing.
  - If they have *different* signs the speed is decreasing.
  - If the velocity graph is moving away from (towards) the t-axis the speed is increasing (decreasing).
- Use a difference quotient to approximate derivative
- Riemann sum approximations
- Units of measure
- Interpret meaning of a definite integral in context of the problem



2. Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

$y - y_1 = m(x - x_1)$ $\quad m = \dfrac{10-0}{3-0} = \dfrac{10}{3}$ $\left(\dfrac{\text{meters}/\text{sec}}{\text{sec}}\right)$

$y - 0 = \frac{10}{3}(x - 0)$

$y = \frac{10}{3}x$

$v(t) = \frac{10}{3}t$ $\quad \left(\frac{m/s}{s}\right)(\text{sec})$

Velocity for Runner A : $V_A(2) = \frac{10}{3} \cdot 2$

$= \boxed{6.667 \text{ m/sec}}$

1pt - $V_A(2)$

Velocity for Runner B : $V_B(2) = \dfrac{24(2)}{2(2)+3}$

$= \boxed{6.857 \text{ m/sec}}$

1pt - $V_B(2)$

Continue problem 2 on page 7.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

$a(t) = v'(t)$ ... ☺

$v_A(t) = \frac{10}{3}t$

$v_A'(t) = a_A(t) = \frac{10}{3}$ acceleration of runner A

$a_A(2) = \boxed{\frac{10}{3}\ m/sec^2}$

1pt - acceleration for Runner A

$v_B(t) = \frac{24t}{2t+3}$

$v_B'(2) = a_B(2) = \boxed{1.469\ m/sec^2}$ acceleration of Runner B

1pt - acceleration for Runner B

no work... use calculator ... ☺

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

$\int |v(t)|\,dt$

area under velocity graph ... ☺

Runner A

Total distance $= \frac{1}{2}(3)(10) + 7(10)$

$= 85$ meters

Runner B

Total distance $= \int_0^{10} \left|\frac{24t}{2t+3}\right| dt$

$= 83.336$ meters

1pt - setup
1pt - answer

1pt - integral
1pt - answer

3. An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

$a(4) = v'(4)$

$\boxed{a(4) = -.524}$

1pt - answer

(b) Consider the following two statements.

means $v'(t) < 0$

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.
Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

means $v(t)$ + $a(t)$ have same signs

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

Statement I is correct b/c

$a(t) = v'(t) < 0$ on $(3, 4.5)$

1pt - I correct w/ reason

Statement II is correct b/c.

$a(t) < 0$ and $v(t) < 0$ on $(3, 4.5)$.

1pt - II correct
1pt - reason for II

Continue problem 3 on page 9.

(c) What is the total distance traveled by the object over the time interval $0 \le t \le 4$?

$\int |v(t)|\,dt$

Total distance $= \int_0^4 \left|\sin\left(\frac{\pi}{3}t\right)\right| dt$

$= 2.387$

1pt - correct limits on $\int |v(t)|\,dt$ or $\int v(t)\,dt$

1pt - $\int |v(t)|\,dt$

1pt - answer

(d) What is the position of the object at time $t = 4$?

initial position $+ \int v(t)$

$x(4) = x(0) + \int_0^4 v(t)\,dt$

$= 2 + \int_0^4 \sin\left(\frac{\pi}{3}t\right) dt$

$= 3.432$

1pt - integral

1pt - answer

3. A particle moves along the y-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}(e^t)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the acceleration of the particle at time $t = 2$.

$a(2) = v'(2)$

$\boxed{a(2) = -.133}$

1pt - answer

(b) Is the speed of the particle $\underbrace{\text{increasing}}_{a(t) + v(t) \text{ same signs}}$ or $\underbrace{\text{decreasing}}_{a(t) + v(t) \text{ different signs}}$ at time $t = 2$? Give a reason for your answer.

$v(2) = -.436$

1pt - answer w/ reason

Speed increasing @ $t = 2$

b/c $v(t) < 0$ and $a(t) < 0$ @ $t = 2$

OR $v(2) < 0$ and $a(2) < 0$

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Continue problem 3 on page 9.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

↳ abs max

$v(t) = 0$

$t = 0.443$ ← only crit #

$v(t)$: $+$ on $(0, .443)$, $-$ after $.443$

Since $v(t) > 0$ on $[0, .443)$ and $v(t) < 0$ on $(.443, \infty)$, no need to check endpts, abs max @ $t = .443$

$\boxed{\text{Particle reaches highest point @ } t = .443}$

1pt - $v(t) = 0$

1pt - crit # 0.443 as possible abs. max

1pt - justify abs max

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

↳ need velocity

initial position + $\int v(t)$

$$y(2) = y(0) + \int_0^2 v(t)\,dt$$

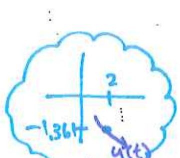
$$= -1 + \int_0^2 \left(1 - \tan^{-1}(e^t)\right) dt$$

$\boxed{y(2) = -1.361}$

1pt - $\int_0^2 v(t)\,dt$

1pt - initial condition

1pt - $y(2)$ correct



$v(2) = -.436$

$v(2) < 0$

$\therefore$, y decreasing

Particle moving away from origin @ $t = 2$ b/c $y(2) < 0$ and $v(2) < 0$

1pt - answer w/reason

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A particle moves along the x-axis so that its velocity v at time t, for $0 \le t \le 5$, is given by $v(t) = \ln\left(t^2 - 3t + 3\right)$. The particle is at position $x = 8$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 4$.

$a(t) = v'(t)$

$a(4) = v'(4)$

$\boxed{a(4) = .714}$

1pt-answer

(b) Find all times t in the open interval $0 < t < 5$ at which the particle changes direction. During which time intervals, for $0 \le t \le 5$, does the particle travel to the left?

↳ v(t) changes signs

↳ negative v(t)

$v(t) = 0$

$t = 1, t = 2$

$v(t)$: + (1) − (2) +

Particle changes direction @ t = 1 and t = 2

b/c v(t) changes signs @ t = 1 and t = 2

1pt - direction changes @ t = 1, t = 2

Particle travels left on (1, 2)

b/c v(t) < 0 on (1, 2)

1pt - interval w/ reason

(c) Find the position of the particle at time $t = 2$.

initial position + $\int v(t)$

$$x(2) = x(0) + \int_0^2 v(t)\,dt$$

$$= 8 + \int_0^2 \ln(t^2 - 3t + 3)\,dt$$

$$\boxed{x(2) = 8.369}$$

1pt - integral
1pt - initial condition x(0)
1pt - answer

distance / time

(d) Find the average speed of the particle over the interval $0 \le t \le 2$.

$$\text{avg speed} = \frac{1}{b-a}\int_a^b |v(t)|\,dt$$

$$= \frac{1}{2}\int_0^2 |v(t)|\,dt$$

$$= .371$$

1pt - integral
1pt - answer

NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

4. A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time t at which the particle is farthest to the left. Justify your answer.

↳ abs. min

$x'(t) = \sin t(e^{-t}\cdot -1) + e^{-t}(\cos t)$ 2pts – $x'(t)$

$= -e^{-t}\sin t + e^{-t}\cos t$

$0 = e^{-t}(-\sin t + \cos t)$ 1pt – $x'(t)=0$

$\cancel{e^{-t}=0}$ $-\sin t + \cos t = 0$

$\cos t = \sin t$

$t = \pi/4,\ 5\pi/4$

check endpts
$x(0) = e^{0}\sin 0 = 0$
$x(2\pi) = e^{-2\pi}\sin 2\pi = 0$

check crit #s
$x(\pi/4) = e^{-\pi/4}\sin \pi/4 = \frac{1}{e^{\pi/4}}\cdot\frac{\sqrt{2}}{2} > 0$
$x(5\pi/4) = e^{-5\pi/4}\sin\frac{5\pi}{4} = \frac{1}{e^{5\pi/4}}\left(-\frac{\sqrt{2}}{2}\right) < 0$

1pt – reason

abs min @ $t = 5\pi/4$

Particle is farthest left when $t = 5\pi/4$

1pt – answer

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Continue problem 4 on page 11.

(b) Find the value of the constant A for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

need $x''(t)$

$x'(t) = e^{-t}(-\sin t + \cos t)$

$x''(t) = (-\sin t + \cos t)(e^{-t} \cdot -1) + e^{-t}(-\cos t - \sin t)$

$= e^{-t}\sin t - e^{-t}\cos t - e^{-t}\cos t - e^{-t}\sin t$

$= -2e^{-t}\cos t$

2pts - $x''(t)$

$Ax''(t) + x'(t) + x(t) = 0$

$A(-2e^{-t}\cos t) + -e^{-t}\sin t + e^{-t}\sin t + e^{-t}\sin t = 0$

1pt - subs in x'', x', x

$-2Ae^{-t}\cos t + e^{-t}\cos t = 0$

$e^{-t}\cos t(-2A + 1) = 0$

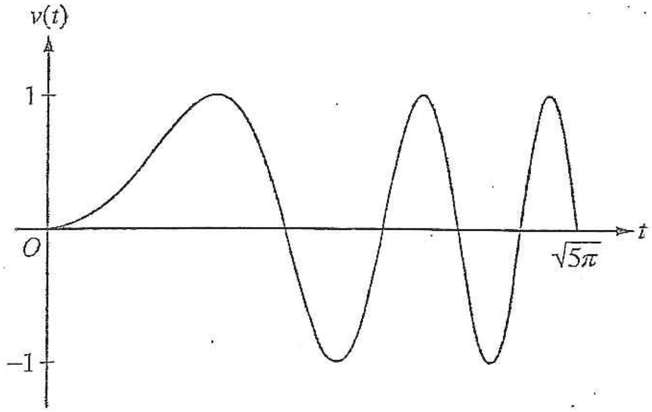
$e^{-t}\cos t = 0$ $\times$

$-2A + 1 = 0$

$-2A = -1$

$\boxed{A = \frac{1}{2}}$

1pt - answer



2. A particle moves along the x-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = \sin(t^2)$. The graph of v is shown above for $0 \leq t \leq \sqrt{5\pi}$. The position of the particle at time t is $x(t)$ and its position at time $t = 0$ is $x(0) = 5$.

(a) Find the acceleration of the particle at time $t = 3$.

$a(t) = v'(t)$

$a(3) = v'(3)$

$\boxed{a(3) = -5.467}$

1pt - a(3)

(b) Find the $\underbrace{\text{total distance}}_{\int |v(t)|\, dt}$ traveled by the particle from time $t = 0$ to $t = 3$.

Total distance $= \int_0^3 |v(t)|\, dt$

$= 1.702$

1pt - setup
1pt - answer

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Continue problem 2 on page 7.

(c) Find the position of the particle at time $t = 3$.

initial position + $\int v(t)$

1pt - integrand
1pt - uses initial condition
1pt - answer

$x(3) = x(0) + \int_0^3 v(t)\,dt$

$= 5 + \int_0^3 v(t)\,dt$

$\boxed{x(3) = 5.774}$

(d) For $0 \le t \le \sqrt{5\pi}$, find the time t at which the particle is farthest to the right. Explain your answer.

need $v(t) = 0$

↳ abs min

rel. min @ $t = 1.772$ and $t = 3.070$ b/c $v(t)$ changes from pos to neg @ these t-values

1pt - $v(t) = 0$

check rel. min's

$x(1.772) = 5 + \int_0^{1.772} v(t)\,dt$

$= 5.895$

$x(3.070) = 5 + \int_0^{3.070} v(t)\,dt$

$= 5.788$

check endpts

$x(0) = 5$

$x(3.963) = 5 + \int_0^{3.963} v(t)\,dt$

$= 5.752$

1pt - reasoning

$\boxed{\text{Particle farthest right when } t = 1.772}$

1pt - answer

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