BC

Topic 1: Particle Motion

Particle moving on a line

These questions may give the position equation, the velocity equation or the acceleration equation along with an initial condition. Students may be asked about the motion of the particle: its direction, when it changes direction, its maximum position in one direction (farthest left or right) etc. Speed, the absolute value of velocity, is also a common topic.

The particle may be a "particle," a person, car, etc. The position, velocity or acceleration may be given as an equation, a graph or a table. There are a lot of different things students may be asked to find. While particles don't really move in this way, the question is a versatile way to test a variety of calculus concepts.

What students should know how to do (AB and BC):

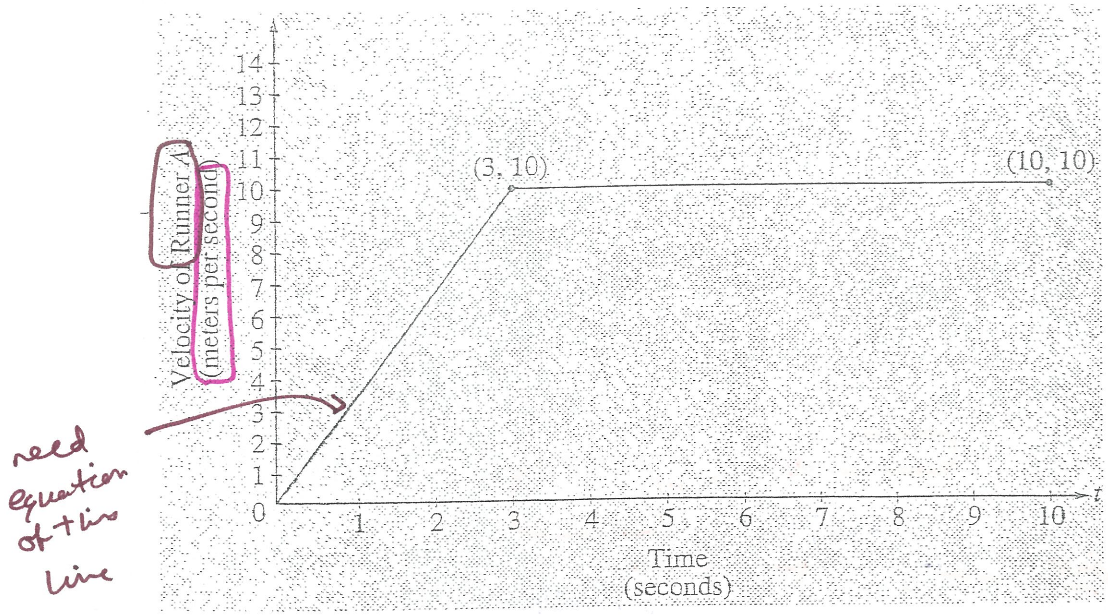
- Initial value differential equation problems: given the velocity or acceleration with initial condition(s) find the position or velocity.
- Distinguish between position at some time (displacement) and the total distance traveled during the time.
  - The total distance traveled is the definite integral of the absolute value of the rate of change (velocity): $\int_a^b |v(t)|\,dt$
  - The net distance (displacement) is the definite integral of the rate of change (velocity): $\int_a^b v(t)\,dt$
  - The final position is the initial position plus the definite integral of the rate of change from $x = a$ to $x = t$: $s(t) = s(a) + \int_a^t v(x)\,dx$ Notice that this is an accumulation function equation.
- Find the speed at a particular time. The speed is the absolute value of the velocity.
- Find average speed, velocity, or acceleration
- Determine if the speed is increasing or decreasing.
  - If at some time, the velocity and acceleration have the *same* sign then the speed is increasing.
  - If they have *different* signs the speed is decreasing.
  - If the velocity graph is moving away from (towards) the t-axis the speed is increasing (decreasing).
- Use a difference quotient to approximate derivative
- Riemann sum approximations
- Units of measure
- Interpret meaning of a definite integral in context of the problem

Particle moving on a Plane (parametric/vector for BC only)

On the BC exam particles often move in the plane. Their position is defined by two parametric equations $x = x(t)$ and $y = y(t)$ or the equivalent vector $\langle x(t), y(t)\rangle$. The velocity is the vector $(x'(t), y'(t))$ and the acceleration is the vector $(x''(t), y''(t))$. Any of these three may be given with initial conditions(s) and student may be asked to find the others.

What students should know how to do:

- Initial value differential equation problems – given the velocity or acceleration with initial conditions, find the position and/or velocity.
- Find the speed at time t: speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$
- Use the definite integral for arc length to find the distance traveled. $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$
- Vectors are given in ordered pair form; answers may be in ordered pairs form or $\overline{i}$-$\overline{j}$ form using parentheses ( ) or pointed brackets <>.



2. Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

Runner A

$y - y_1 = m(x - x_1)$

$m = \dfrac{10-0}{3-0}$ $\dfrac{m/sec}{sec}$

$m = \dfrac{10}{3}$

$y - 0 = \dfrac{10}{3}(x - 0)$

$y = \dfrac{10}{3}x$

$v(t) = \dfrac{10}{3}t$

$v_A(2) = \dfrac{10}{3} \cdot 2$ $\left(\dfrac{\frac{m}{sec}}{\cancel{sec}}\right)(\cancel{sec})$

$= 6.667\ m/sec$

Runner B

$v_B(2) = \dfrac{24(2)}{2(2)+3}$

$= 6.857\ m/sec$

1pt : $v_A(2)$

1pt : $v_B(2)$

Continue problem 2 on page 7.

$\rightarrow a(t) = v'(t)$

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

Acceleration Runner A

$v'_A(t) = a_A(t)$

$= \frac{10}{3}$

$\boxed{a_A(2) = \frac{10}{3} \text{ m/sec}^2}$

Acceleration Runner B

$v'_B(2) = a_B(2)$

$\boxed{a_B(2) = 1.469 \text{ m/sec}^2}$

1pt: acceleration Runner A

1pt: acc Runner B

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

$\rightarrow \int |v(t)|\,dt$

area under velocity graph

Runner A

Total Distance $= \frac{1}{2}(3)(10) + 7(10)$

$= 85$ meters

Runner B

Total Distance $= \int_0^{10} \left| \frac{24t}{2t+3} \right| dt$

$= 83.336$ meters

1pt: set up

1pt: answer

1pt: integral

1pt: answer

1pt – units correct in part a, b, and c.

GO ON TO THE NEXT PAGE.

3. An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

$a(4) = v'(4)$

$a(4) = -.524$

1pt : answer

(b) Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing. means $v'(t) < 0$

Statement II: For $3 < t < 4.5$, the speed of the object is increasing. $\rightarrow$ means $a(t)$ + $v(t)$ have same signs

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

Statement I is correct b/c

$a(t) = v'(t) < 0$ on $(3, 4.5)$

1pt : I correct w/ reason

Statement II is correct b/c

$a(t) < 0$ and $v(t) < 0$ on $(3, 4.5)$

1pt : II correct

1pt : reason for II

Continue problem 3 on page 9.

(c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

↳ $\int |v(t)|\, dt$

Total distance $= \int_0^4 |\sin(\frac{\pi}{3}t)|\, dt$

$= 2.387$

1pt: correct limits on $\int |v(t)|\, dt$ or $\int v(t)\, dt$

1pt: $\int |v(t)|\, dt$

1pt: answer

(d) What is the position of the object at time $t = 4$?

↳ initial position $+ \int v(t)$

$x(4) = x(0) + \int_0^4 v(t)\, dt$

$= 2 + \int_0^4 v(t)\, dt$

$= 3.432$

1pt: integral

1pt: answer

3. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1+t^3}$$

initial y-value

for $t \geq 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

$\hookrightarrow a(t) = v'(t)$ $\hookrightarrow |v(t)| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$

$v(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$

$a(t) = v'(t)$

$\boxed{a(2) = \langle .396, -.741 \rangle}$

1pt: acceleration @ $t=2$

speed = $|v(t)|$

$|v(2)| = \sqrt{(x'(2))^2 + (y'(2))^2}$

$\boxed{|v(2)| = 1.208}$

1pt: speed @ $t=2$

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

$\hookrightarrow$ dy/dx DNE, $\frac{dy/dt}{dx/dt} = \frac{\#}{0}$

so, $dx/dt = 0$

$\sin^{-1}(1 - 2e^{-t}) = 0$

$\boxed{t = 0.693}$

1pt: $x'(t) = 0$

1pt: answer

Continue problem 3 on page 9

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t \to \infty} m(t)$.

$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$m(t) = \frac{\frac{4t}{1+t^2}}{\sin^{-1}(1-2e^{-t})}$$

1pt: m(t)

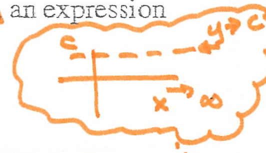
$$\lim_{t \to \infty} m(t) = \frac{0}{\sin^{-1}(1)}$$

$$= 0$$

1pt: limit value

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

$\lim_{x \to \infty} f(x) = c$



$\lim_{t \to \infty} x(t) = \infty$ $\qquad$ $\lim_{t \to \infty} y(t) = c$

$\lim_{t \to \infty} x(t) = \infty$,

$$\lim_{t \to \infty} y(t) = \text{initial y-value} + \int_2^{\infty} y'(t)\,dt$$

$$\boxed{c = -3 + \int_2^{\infty} \frac{4t}{1+t^3}\,dt}$$

1pt: integrand

1pt: limits

1pt: initial value

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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CALCULUS AB
SECTION II, Part A
Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.

1. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t} \text{ and } \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right).$$

The particle is at position $(1, 5)$ at time $t = 4$.

(a) Find the acceleration vector at time $t = 4$.

$a(t) = v'(t)$

$a(4) = \langle x''(4), y''(4) \rangle$

$\boxed{a(4) = \langle .433, -11.872 \rangle}$

1 pt: answer

(b) Find the y-coordinate of the position of the particle at time $t = 0$.

initial y-value $+ \int y'(t)\,dt$

$y(0) = y(4) + \int_4^0 3\cos\left(\frac{t^2}{2}\right) dt$

$= 5 + -3.399$

$\boxed{y(0) = 1.601}$

1pt: integrand

1pt: initial condition $y(4) = 5$

1pt: answer

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Continue problem 1 on page 5.

(c) On the interval $0 \le t \le 4$, at what time does the speed of the particle first reach 3.5 ?

$\hookrightarrow |v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$$\text{speed} = |v(t)| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$$

1pt: expression for speed

$$3.5 = \sqrt{(\sqrt{3t})^2 + (3\cos(t^2/2))^2}$$

1pt: equation

$$t = 2.226$$

1pt: answer

Speed of particle 1st reaches 3.5 @ $t = 2.226$

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le 4$.

$\hookrightarrow \int |v(t)|\, dt$

$$\text{total distance} = \int_0^4 \sqrt{(\sqrt{3t})^2 + (3\cos(t^2/2))^2}\, dt$$

1pt: integral

1pt: answer

$$= 13.182$$

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CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

4. A particle moves in the xy-plane so that the position of the particle at any time t is given by

$$x(t) = 2e^{3t} + e^{-7t} \text{ and } y(t) = 3e^{3t} - e^{-2t}.$$

(a) Find the velocity vector for the particle in terms of t, and find the speed of the particle at time $t = 0$.

$\hookrightarrow \langle x'(t), y'(t) \rangle$

$\hookrightarrow |v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$$v(t) = \langle x'(t), y'(t) \rangle$$
$$= \langle 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t} \rangle$$

1pt: $x'(t)$
1pt: $y'(t)$

$$\text{speed} = |v(0)| = \sqrt{(6e^{3(0)} - 7e^{-7(0)})^2 + (9e^{3(0)} + 2e^{-2(0)})^2}$$
$$= \sqrt{1^2 + 11^2}$$
$$= \sqrt{122}$$

1pt: speed

(b) Find $\frac{dy}{dx}$ in terms of t, and find $\lim_{t \to \infty} \frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$$

1pt: $\frac{dy}{dx}$ in terms of t

$$\lim_{t\to\infty} \frac{dy}{dx} = \lim_{t\to\infty} \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}}$$
$$= \frac{9}{6} \quad \leftarrow \text{ok to stop here}$$
$$= 3/2$$

1pt: limit

Continue problem 4 on page 11.

(c) Find each value t at which the line tangent to the path of the particle is horizontal, or explain why none exists.

$\rightarrow dy/dx = 0$, $y'(t) = 0$

$y'(t) = 9e^{3t} + 2e^{-2t}$

$9e^{3t} + 2e^{-2t} > 0 \quad \forall t,$

1pt: considers $y'(t) = 0$

$\therefore$, tangent line will never be horizontal.
($y'(t)$ will never equal zero)

1pt: explain why none exists

so, none exists

(d) Find each value t at which the line tangent to the path of the particle is vertical, or explain why none exists.

$\rightarrow \frac{dy}{dx} = \frac{1}{0} \rightarrow \frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$x'(t) = 6e^{3t} - 7e^{-7t}$

$6e^{3t} - 7e^{-7t} = 0$

$6e^{3t} = 7e^{-7t}$

$e^{10t} = \frac{7}{6}$

1pt: considers $x'(t) = 0$

$10t = \ln\left(\frac{7}{6}\right)$

$t = \frac{1}{10}\ln\left(\frac{7}{6}\right)$

1pt: solution

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