AP Topic: Area & Volume (BC)

Given equations that define a region in the plane students are asked to find its area and the volume of the solid formed when the region is revolved around a line or used as a base of a solid with regular cross-sections. This standard application of the integral has appeared every year since 1969 on the AB exam and all but one year on the BC exam.

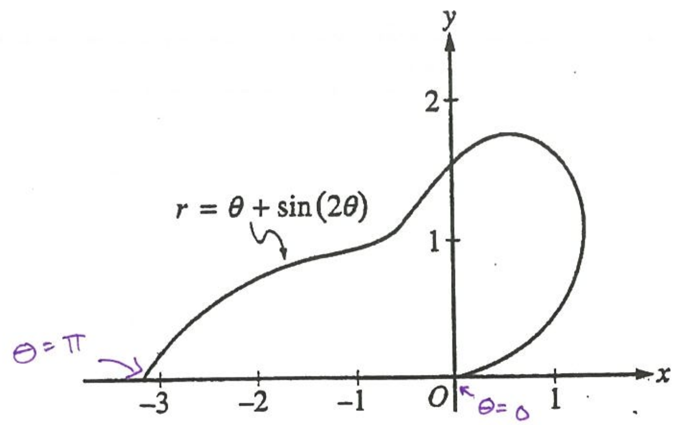
<u>If this appears on the calculator active section</u>: It is expected that the definite integrals will be evaluated on a calculator. Students should write the definite integral with limits on their paper and put its value after it. It is *not* required to give the antiderivative and if students give an incorrect antiderivative they will lose credit even if the final answer is (somehow) correct.

There is a calculator program available that will give the set-up and not just the answer so recently this question has been on the no calculator section. (The good news is that the integrals will be easy or they will be set-up but do not integrate questions.)

What students should know how to do:

- Find the intersection(s) of the graphs and use them as limits of integration (calculator equation solving). Write the equation followed by the solution; showing work is not required. Usually no credit until the solution is used in context.
- Find the area of the region between the graph and the x-axis or between two graphs.
- Find the volume when the region is revolved around a line, not necessarily and axis, by the disk/washer method. (Shell method is *never necessary* but is eligible for full credit if properly used).
- Find the volume of a solid with regular cross-sections whose base is the region between the curves. But see 2009 AB 4(b)
- Find the equation of a vertical line that divides the region in half (area or volume). This involves setting up and solving an integral equation where the limit is the variable for which the equation is solved.

- For BC only – find the area of a region bounded by polar curves.
  $A=\frac{1}{2}\int_{\theta_1}^{\theta_2}(r(\theta))^2\,d\theta$



The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $\boxed{0 \le \theta \le \pi,}$ where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

(a) Find the area bounded by the curve and the x-axis.

$$\text{Area} = \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (\theta + \sin 2\theta)^2 \, d\theta$$

$$= 4.382$$

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

$\hookrightarrow x = r\cos\theta$

$$-2 = (\theta + \sin 2\theta)\cos\theta$$

$$\theta = 2.786$$

Continue problem 2 on page 7.

For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

$\frac{dr}{d\theta} < 0 \rightarrow r$ dec

on $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$, $\frac{dr}{d\theta} < 0$, $\therefore$ r is dec on $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$.

This means that r is getting closer to the origin.

r is radius
↳ length from center (origin)

Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

↳ abs max of r → candidates test!

$\frac{dr}{d\theta} = 0$

$1 + 2\cos 2\theta = 0$

$\cos 2\theta = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

$\theta = \frac{\pi}{3}, \cancel{\frac{2\pi}{3}}$ ← not in interval $[0, \frac{\pi}{2}]$

$r = \theta + \sin 2\theta$

$r(0) = 0$

$r(\frac{\pi}{3}) = 1.913$

$r(\frac{\pi}{2}) = 1.571$

The greatest distance from the origin occurs @ $\theta = \frac{\pi}{3}$.

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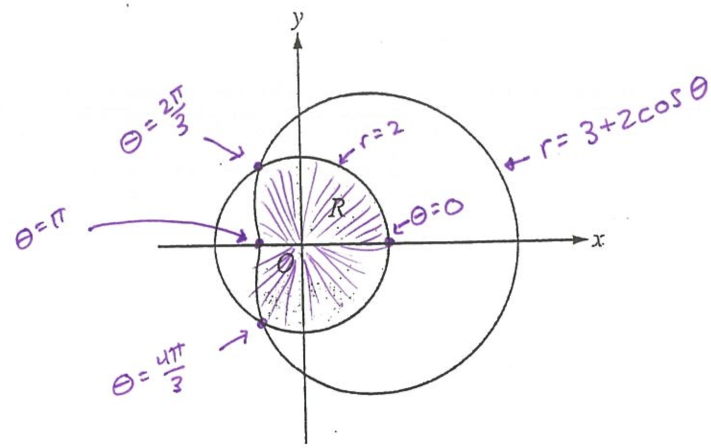

3. The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R. $\rightarrow$ $\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$ ... ☺

$$\text{Area} = 2\left(\frac{1}{2}\int_0^{\frac{2\pi}{3}} 2^2\,d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (3+2\cos\theta)^2\,d\theta\right)$$

$$= 10.370$$

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Continue problem 3 on page 9.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\boxed{\frac{dr}{dt} = \frac{dr}{d\theta}}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

→ move toward or away from origin?

$r = 3 + 2\cos\theta$

$\left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = \left.\frac{dr}{dt}\right|_{\theta=\pi/3} = -1.732$

$r(\frac{\pi}{3}) = 3 + 2\cos\frac{\pi}{3}$

> 0

Particle moving closer to origin b/c $r(\frac{\pi}{3}) > 0$ and $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} < 0$

(c) For the particle described in part (b), $\boxed{\frac{dy}{dt} = \frac{dy}{d\theta}}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

→ move toward or away from ~~origin~~ x-axis?

$y = r\sin\theta$

$y = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

so, $\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = 0.5$

$y(\frac{\pi}{3}) = (3 + 2\cos\frac{\pi}{3})\sin\frac{\pi}{3}$

> 0

Particle is moving away from x-axis b/c $y(\frac{\pi}{3}) > 0$ and $\left.\frac{dy}{dt}\right|_{\theta=\pi/3} < 0$

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END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

NO CALCULATOR ALLOWED

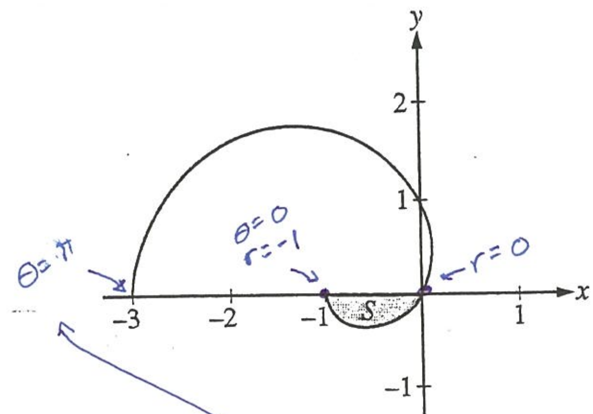
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



4. The graph of the polar curve $r = 1 - 2\cos\theta$ for $\boxed{0 \le \theta \le \pi}$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

   (a) Write an integral expression for the area of S.

$1 - 2\cos\theta = -1$

$2 = 2\cos\theta$

$1 = \cos\theta$

$0 = \theta$

$1 - 2\cos\theta = 0$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2 \, d\theta$$

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Continue problem 4 on page 11.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

$x = r\cos\theta$

$x = (1 - 2\cos\theta)\cos\theta$

$\frac{dx}{d\theta} = \cos\theta(2\sin\theta) + (1 - 2\cos\theta)(-\sin\theta)$ ← ok to stop here for AP

$= 2\sin\theta\cos\theta - \sin\theta + 2\sin\theta\cos\theta$

$= 4\sin\theta\cos\theta - \sin\theta$

$y = r\sin\theta \rightarrow y = (1 - 2\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = \sin\theta(2\sin\theta) + (1 - 2\cos\theta)(\cos\theta)$ ← ok to stop here for AP

$= 2\sin^2\theta + \cos\theta - 2\cos^2\theta$

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

$y - y_1 = m(x - x_1)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$x = (1 - 2\cos\theta)\cos\theta$

$x(\frac{\pi}{2}) = (1 - 2\cos\frac{\pi}{2})\cos\frac{\pi}{2}$

$x(\frac{\pi}{2}) = 0$

$y = (1 - 2\cos\theta)\sin\theta$

$y(\frac{\pi}{2}) = (1 - 2\cos\frac{\pi}{2})\sin\frac{\pi}{2}$

$y(\frac{\pi}{2}) = 1$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{\left.dy/d\theta\right|_{\theta=\frac{\pi}{2}}}{\left.dx/d\theta\right|_{\theta=\frac{\pi}{2}}} = \frac{2(\sin\frac{\pi}{2})^2 + \cos\frac{\pi}{2} - 2(\cos\frac{\pi}{2})^2}{4\sin\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}} = \frac{2}{-1} = -2$$

$\boxed{y - 1 = -2(x - 0)}$

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