Topic 3: Modeling Rates

The integral of a rate of change gives the amount of change (FTC). The general form of the equation is $f(x)=f(x_0)+\int_{x_0}^{x} f'(t)\,dt$, $x=x_0$ is the initial time, and $f(x_0)$ is the initial value. Since this is one of the main interpretations of the definite integral the concept may come up in a variety of situations.

What students should know how to do?

- Understand the question. It is often not necessary to as much computation as it seems at first.
- The FTC may help differentiating *F*.
- Often these problems contain a lot of writing; be ready to read and apply; recognize that rate = derivative.
- Recognize a rate from the units given without the words "rate" or "derivative."
- Explain the meaning of a derivative or definite integral or its value in terms of the context of the problem.
- In-out problems: 2 rates of change work together but in opposite directions.
- Max/min and inc/dec analysis.
- Explain the meaning of a definite integral in context. The explanation should include (1) what the integral gives, (2) the units and (3) an accounting of the limits of integration.

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CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

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4. Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

derivative

→ starting water

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?

water leaked out $= \int_0^3 \sqrt{t+1}\,dt$ $\quad u = t+1 \quad du = dt \quad u(3) = 4 \quad u(0) = 1$

1pt- limits

1pt- integrand

$= \int_1^4 \sqrt{u}\,du$

$= \int_1^4 u^{1/2}\,du$

$= \frac{2}{3}u^{3/2}\Big|_1^4$

$= \frac{2}{3}(4^{3/2} - 1^{3/2})$ ← ok to stop here ☺

$= \frac{2}{3}(8-1) = \boxed{\frac{14}{3}}$

1pt - answer

(b) How many gallons of water are in the tank at time $t = 3$ minutes?

Water in tank = water start with + water pumped in − water leaked out

$= 30 + 8(3) - \frac{14}{3}$

↖ time 3 minutes

↖ answer from (a)

ok to stop here ☺

1pt - answer

$= \frac{148}{3}$

(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

A(t) = Water start with + water pumped in − water leaked out

$A(t) = 30 + 8t - \int_0^t \sqrt{x+1}\,dx$

1pt : $30+8t$

1pt : $-\int_0^t \sqrt{x+1}\,dx$

(d) At what time t, for $0 \leq t \leq 120$, is the amount of water in the tank a maximum? Justify your answer.

↳ $A'=0$ + A' changes from ~~neg to pos~~ pos to neg

$A'(t) = \frac{d}{dt}\left(30 + 8t - \int_0^t \sqrt{x+1}\,dx\right)$

$= 8 - \sqrt{t+1}$

1pt : $A'(t) = 0$

$8 - \sqrt{t+1} = 0$

$8 = \sqrt{t+1}$

$64 = t+1$

$63 = t$

1pt : solves for t

1pt : justification

Water in tank is max @ $t=63$ b/c $A'(t)$ pos on $(0,63)$ and $A'(t)$ neg on $(63,120)$

so, no need to check endpts ... ☺

CALCULUS ~~BC~~AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.

1. Traffic flow is defined as the rate (derivative) at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$\text{rate} \longrightarrow F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

$$\#\text{ cars pass the intersection} = \int_0^{30} F(t)\,dt$$

$$= 2474 \text{ cars}$$

1pt: limits
1pt: integrand
1pt: answer

(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

$F(t)$ inc? $F'(t) > 0$ $F(t)$ dec? $F'(t) < 0$

$F'(7) = -1.873$

The traffic flow is decreasing @ $t = 7$

b/c $F'(7) < 0$

1pt = answer w/ reason

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Continue problem 1 on page 5.

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(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

F(t)

$\frac{1}{b-a}\int_a^b$

$\frac{1}{15-10}\int_{10}^{15} F(t)\,dt$

$\left(\frac{1}{\text{min}}\right)(\text{cars})$

$$\text{avg value} = \frac{1}{15-10}\int_{10}^{15} F(t)\,dt$$

$$= 81.899 \text{ cars/min}$$

1pt: units
1pt: integrand
1pt: answer

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

F(t)

$\frac{f(b)-f(a)}{b-a}$

$$\text{avg rate of change} = \frac{F(15)-F(10)}{15-10} \qquad \frac{\text{cars/min}}{\text{min}} \; ☺$$

$$= 1.518 \text{ cars/min}^2$$

1pt: answer

1pt: correct units in (c) and (d)

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2. For $0 \leq t \leq 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos\left(\frac{t}{5}\right)$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.

this is rate →

↳ initial mosquitoes

(a) Show that the number of mosquitoes is increasing at time $t = 6$.

↳ rate of mosquitoes > 0

$R(6) = 4.438$

The # of mosquitoes is inc @ $t=6$

b/c $R(6) > 0$

1pt: shows $R(6)>0$

(b) At time $t = 6$, is the number of mosquitoes increasing at an increasing rate, or is the number of mosquitoes increasing at a decreasing rate? Give a reason for your answer.

↳ need $R'(6) > 0$

↳ need $R'(6) < 0$

1pt: finds $R'(6)$

$R'(6) = -1.913$

The # of mosquitoes is inc @ a dec. rate

@ $t = 6$

b/c $R'(6) < 0$

1pt: answer w/ reason

Continue problem 2 on page 7.

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(c) According to the model, how many mosquitoes will be on the island at time $t = 31$? Round your answer to the nearest whole number.

\# mosquitoes on island = initial mosquitoes + additional mosquitoes

$= 1000 + \int_0^{31} R(t)\,dt$

1pt: integral

$= 964.335$

$\boxed{964 \text{ mosquitoes}}$

1pt: answer

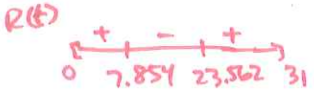
(d) To the nearest whole number, what is the maximum number of mosquitoes for $0 \le t \le 31$? Show the analysis that leads to your conclusion.

↳ abs max ... get rel max + endpts.

$R(t) = 0$

@ $t = 0$, $t = 7.854$, $t = 23.562$

1pt: crit #s



Rel max @ $t = 7.854$ b/c R(t) changes from pos to neg @ $t = 7.854$

1pt: analysis of checking crit #s & end pts

check endpts { @ $t = 0$, $1000 + \int_0^0 R(t)\,dt = 1000$

@ $t = 31$, $1000 + \int_0^{31} R(t)\,dt = 964$ }

check rel. max { @ $t = 7.854$, $1000 + \int_0^{7.854} R(t)\,dt = 1039.357$

1pt: integral

$\boxed{\text{max \# of mosquitoes is } 1039}$

1pt: answer

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2. The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

rate remove → $$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

rate added → $$S(t) = \frac{15t}{1+3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

amount sand removed $= \int_0^6 R(t)\,dt$

$= 31.816\ yd^3$

$\left(\frac{\text{cubic yds}}{\text{hr}}\right)(\text{hr})$ ☺

1pt: integral

1pt: answer w/ units

(b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

sand on the beah = initial sand + sand added − sand removed

$$Y(t) = 2500 + \int_0^t [S(x) - R(x)]\,dx$$

1pt: integrand

1pt: limits

1pt: answer

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(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

rate sand on beach $= y'(t)$

$$y'(t) = \frac{d}{dt}\left(2500 + \int_0^t (S(x) - R(x))\,dx\right)$$

$$= S(t) - R(t)$$

$$y'(4) = S(4) - R(4)$$

$$= -1.909 \ \text{yd}^3/\text{hr}$$

1pt: answer (units not required)

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

abs. min

$y'(t) = 0$ crit #s & check end pts

$y'(t) = 0$

$S(t) - R(t) = 0$ $\quad$ $\boxed{S(t) = R(t)}$ ... ☺

$t = 5.118$

1pt: $y'(t) = 0$

1pt: crit #

check crit # & end pts

$y(0) = 2500$

$$y(5.118) = 2500 + \int_0^{5.118} (S(x) - R(x))\,dx$$

$$= 2492.369$$

$$y(6) = 2500 + \int_0^{6} (S(x) - R(x))\,dx$$

$$= 2493.277$$

Amount of sand min @ $t = 5.118$ hrs

min value is $2492.369 \ \text{yd}^3$

1pt: answer w/ justification

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2. A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$\text{rate in} \rightarrow W(t) = 95\sqrt{t}\,\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$\text{rate out} \rightarrow R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

~~rate of~~ amount of water > 0

rate of amount water = rate in − rate out

$= W(t) - R(t)$

$W(15) - R(15) = -121.09$

Amount of water not inc @ $t = 15$ b/c $W(15) - R(15) < 0$

1pt: answer w/ reason

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

Amount water in tank = initial amount + $\int$ (rate in − rate out) dt

$$= 1200 + \int_0^{18} (W(t) - R(t))\,dt$$

1pt: limits
1pt: integrand

$$= 1309.788$$

$$\boxed{1310 \text{ gallons}}$$

1pt: answer

Continue problem 2 on page 7.

(c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

rate of water = 0

check crit (rel min) + end pts

$W(t) - R(t) = 0$

$t = 0,\ t = 6.49$[illegible]$,\ t = 12.975$

1pt: crit #s

check crit #s & end pts

$@\ t=0, \quad 1200 + \int_0^0 (W(t) - R(t))\,dt = 1200$

$@\ t=6.495, \quad 1200 + \int_0^{6.495} (W(t) - R(t))\,dt = 525.242$

$@\ t=12.975, \quad 1200 + \int_0^{12.975} (W(t) - R(t))\,dt = 1697.441$

$@\ t=18, \quad 1200 + \int_0^{18} (W(t) - R(t))\,dt = 1309.788$

1pt: analysis for abs min

Amount of water is abs min @ $t = 6.495$

1pt: $t = 6.495$

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

no water in tank ⇒ = 0

@ $t = 18$, amount of water = 1310 gallons

water in tank − water removed = 0

$$\boxed{1309.788 - \int_{18}^{k} R(t)\,dt = 0}$$

1pt: limits

1pt: equation

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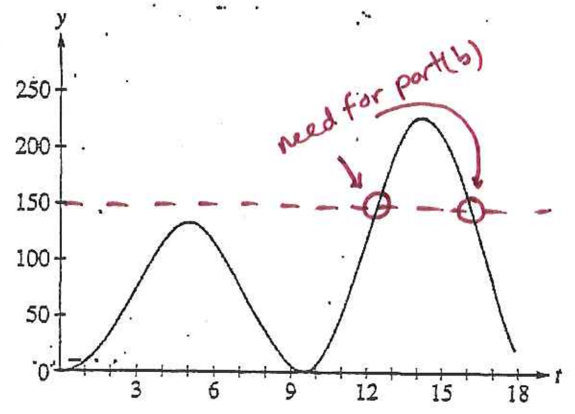

2. At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.

(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

↳ ∫ rate left

$$\#\text{ cars turn left} = \int_0^{18} L(t)\,dt$$

$$= 1657.824$$

$$= \boxed{1658 \text{ cars}}$$

1pt: setup
1pt: answer

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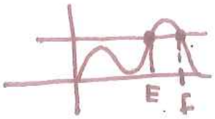
(b) Traffic engineers will consider turn restrictions when $L(t) \geq 150$ cars per hour. Find all values of t for which $\underbrace{L(t) \geq 150}$ and compute the average value of L over this time interval. Indicate units of measure.

↓

$\frac{1}{b-a}\int_a^b \rightarrow \frac{1}{hr}(\text{cars})$

$L(t) \geq 150$ on $[12.428, 16.122]$

1pt: t-interval on which $L(t) \geq 150$



$$\text{avg value} = \frac{1}{F-E}\int_E^F L(t)\,dt$$

$E = 12.428$

$F = 16.122$

1pt: avg value integral

$$= 199.426 \text{ cars/hr}$$

1pt: answer w/ units

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

(# cars left)(# oncoming cars) > 200000

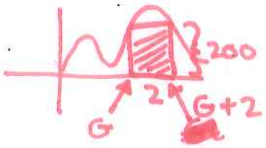
(#cars left)(500) > 200000

cars left > 400

1pt: considers 400 cars

need, $\int_G^{G+2} L(t)\,dt > 400$

1pt: valid interval $[h, h+2]$



use $\int_{13.5}^{15.5} L(t)\,dt = 431.931$

1pt: value of integral $\int_h^{h+2} L(t)\,dt$

Since $\int_{13.5}^{15.5} L(t)\,dt > 400$, traffic signal is required

1pt: answer + explanation

$\int L(t)\,dt$ can have any limit #s where $\int L(t)\,dt > 400$ for any 2-hour interval ...

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3. The wind chill is the temperature, in degrees Fahrenheit (°F), a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is 32°F, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

(a) Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

$$W'(20) = W'(v)\Big|_{v=20}$$

$$= -.286\ {}^\circ F/mph$$

1pt: value

☺

$W' < 0 \Rightarrow W$ is dec

The wind chill temp is decreasing @ 0.286 °F/mph when velocity of wind is 20mph

1pt: explanation

(b) Find the average rate of change of W over the interval $5 \le v \le 60$. Find the value of v at which the instantaneous rate of change of W is equal to the average rate of change of W over the interval $5 \le v \le 60$.

$\frac{f(b)-f(a)}{b-a}$

↳ $W'(v)$

↳ $\frac{W(60)-W(5)}{60-5}$

$$\text{avg rate of change} = \frac{W(60)-W(5)}{60-5}$$

1pt: avg rate of change

$$= -.254$$

$$W'(v) = \text{avg rate of change}$$

1pt: $W'(v)$ = avg rate of change

$$W'(v) = -.254$$

$$v = 23.011 \text{ mph}$$

1pt: answer

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(c) Over the time interval $0 \le t \le 4$ hours, the air temperature is a constant 32°F. At time $t = 0$, the wind velocity is $v = 20$ mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at $t = 3$ hours? Indicate units of measure.

$\hookrightarrow \frac{dw}{dt}$

$\frac{dv}{dt}$

$@ t = 0, \; v(0) = 20 \; m/hr$

$\frac{dv}{dt} = 5 \; m/hr/hr$

1pt: $\frac{dv}{dt} = 5$

$\frac{dw}{dt} = ? \quad @ \; t = 3$

$W = 55.6 - 22.1v^{.16}$

$\frac{dw}{dt} = -22.1(.16v^{.16-1})\frac{dv}{dt}$

$\frac{dw}{dt}\Big|_{t=3} = -22.1(.16(35)^{.84})(5)$

$\frac{dv}{dt} = 5 \; m/hr/hr$

$v(t) = 5t + C$

$v(0) = 5(0) + C$

$20 = C$

$v(t) = 5t + 20$

$v(3) = 5(3) + 20$

$v(3) = 35$

1pt: uses v(3)

$\frac{dw}{dt}\Big|_{t=3} = -.892 \; °F/hr$

1pt: answer

1pt: unit in (a) + (c) are correct

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END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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