Topic 4: Modeling Rates Analytically & Numerically

The integral of a rate of change gives the amount of change (FTC). The general form of the equation is $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt$, $x = x_0$ is the initial time, and $f(x_0)$ is the initial value. Since this is one of the main interpretations of the definite integral the concept may come up in a variety of situations.

What students should know how to do?

- Understand the question. It is often not necessary to as much computation as it seems at first.
- The FTC may help differentiating *F*.
- Often these problems contain a lot of writing; be ready to read and apply; recognize that rate = derivative.
- Recognize a rate from the units given without the words "rate" or "derivative."
- Explain the meaning of a derivative or definite integral or its value in terms of the context of the problem.
- In-out problems: 2 rates of change work together but in opposite directions.
- Max/min and inc/dec analysis.
- Explain the meaning of a definite integral in context. The explanation should include (1) what the integral gives, (2) the units and (3) an accounting of the limits of integration.

Tables may be used to test a variety of ideas in calculus including analysis of functions, accumulation, position-velocity-acceleration, *et al.*

What students should be able to do:

- Approximate the derivative using a difference quotient.
- Use Riemann sums (left, right, midpoint) or a trapezoidal approximation to approximate the value of a definite integral using values in the table (typically with uneven subintervals). (Trapezoidal Rule, *per se*, is not required.
- Average value and the MVT may appear
- Questions about the Rolle's theorem, MVT, IVT, etc.

Do not: Use a calculator to find a regression equation and then use that to answer parts of the question. (While finding them ids perfectly good mathematics, regression equations are not one of the four things students may do with their calculator and give only an approximation of our function.)

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4 → mdpt
6	10.8
9	11.2 → mdpt
12	11.4
15	11.3 → mdpt
18	10.7
21	10.2 → mdpt
24	9.6

rate

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

(a) Use a midpoint Riemann sum with $\boxed{\text{4 subdivisions of equal}}$ length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

∫ rate

$\Delta t = \frac{24-0}{4} = 6$

$\int_0^{24} R(t)\,dt \approx 6(10.4) + 6(11.2) + 6(11.3) + 6(10.2)$

$= 258.6$ gallons

1pt: $R(3)+R(9)+R(15)+R(21)$

1pt: answer

$\int_0^{24} R(t)\,dt$ is the total amount of water that flows out of pipe in gallons from $t=0$ to 24 hours

1pt: explanation w/units

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

MVT → $f'(x) = \frac{f(b)-f(a)}{b-a}$

$\left.\begin{array}{l} R(0)=9.6 \\ R(24)=9.6 \end{array}\right\} \frac{R(24)-R(0)}{24-0} = 0$

Since R is a diff'able function, MVT guarantees there is some time on $(0, 24)$ s.t. $R'(t)=0$ since $\frac{R(24)-R(0)}{24-0} = 0$.

1pt: answer

1pt: MVT explanation

Continue problem 3 on page 9.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$.
Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period.
Indicate units of measure.

avg value of Q(t)

1pt: limits & constant

$$\text{avg value of } Q(t) = \frac{1}{24-0}\int_0^{24} Q(t)\,dt$$

1pt: Q(t) as integrand

$$= 10.785 \text{ gallons/hr}$$

1pt: answer

1pt: correct units in part (a) & (c)

$\Delta x=1$ $\Delta x=4$ $\Delta x=1$ $\Delta x=2$

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

3. A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $\overset{\text{slope}}{T'(7)}$. Show the work that leads to your answer. Indicate units of measure.

$$T'(7) = \frac{T(8)-T(6)}{8-6}$$

$$= \frac{55-62}{2} \quad \frac{°C}{min}$$

$$= -3.5\ °C/min$$

1pt - answer w/ work

Work for problem 3(b)

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a $\boxed{\text{trapezoidal}}$ sum with the four subintervals indicated by the data in the table. Indicate units of measure.

avg value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$\frac{1}{2}(b_1+b_2)(h)$

not equal !!

$$\text{avg temp of wire} = \frac{1}{8-0}\int_0^8 T(x)\,dx$$

$$= \frac{1}{8}\int_0^8 T(x)\,dx$$

1pt: $\frac{1}{8}\int_0^8 T(x)\,dx$

1pt: trap sum

1pt: answer

$$\frac{1}{8}\int_0^8 T(x)\,dx = \frac{1}{8}\Big[\tfrac{1}{2}(100+93)(1) + \tfrac{1}{2}(93+70)(4) + \tfrac{1}{2}(70+62)(1) + \tfrac{1}{2}(62+55)(2)\Big]$$

$$= 75.688\ °C$$

$\frac{1}{cm}(\ °C\ )(cm) = °C$

Continue problem 3 on page 9.

Work for problem 3(c)

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

$\int_0^8 T'(x)\,dx = T(8) - T(0)$

$= 55 - 100$

$= -45°\,C$

1pt: answer

1pt: meaning w/ units

$\int_0^8 T'(x)\,dx$ means the temperature of wire decreases 45°C from 0 to 8 cm from heated end of wire.

Work for problem 3(d)

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

By MVT, since T is twice diff'able, T' is cont + diff'able,

$T''(c) = \dfrac{T'(x_1) - T'(x_2)}{x_1 - x_2}$

1pt: calculates 2 slopes

$T'(x_1)$ on $[1,5] \rightarrow \dfrac{93-70}{1-5} = -5.75$

$T'(x_2)$ on $[5,6] \rightarrow \dfrac{70-62}{5-6} = -8$

So, T' is decreasing somewhere on [1,6]

1pt - answer w/ explanation

∴, T'' is not always positive on [0,8]

So data not consistent w/ $T''(x) > 0$ on [0,8]

1pt - correct units in (a), (b), + (c)

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

mdpt mdpt mdpt

4. Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

$\hookrightarrow a(t) = v'(t)$

$$\text{avg acc} = \frac{v(80) - v(0)}{80 - 0} = \frac{49-5}{80} = 0.55 \text{ ft/sec}^2$$

$\frac{\text{ft/sec}}{\text{sec}}$ ... ☺

1pt: answer w/ work

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

$\rightarrow \Delta t = \frac{70-10}{3} = \frac{60}{3} = 20$

$$\int_{10}^{70} v(t)\,dt = 20(22) + 20(35) + 20(44) = 2020 \text{ ft}$$

(sec)(ft/sec) ... ☺

1pt: mdpt sum

1pt: answer

$\int_{10}^{70} v(t)\,dt$ is the distance in ft that the rocket A traveled from $t = 10$ to 70 seconds

1pt: explanation w/ units

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Continue problem 4 on page 11.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

↳ need velocity
$\int a(t) = v(t)$

Rocket B

$$v_B(80) = v(0) + \int_0^{80} a(t)\,dt$$
$$= 2 + \int_0^{80} 3(t+1)^{-1/2}\,dt$$
$$= 2 + \int_1^{81} 3u^{-1/2}\,dt$$

$u = t+1$
$du = dt$
$u(80) = 81$
$u(0) = 1$

$$= 2 + 3(2u^{1/2})\Big|_1^{81}$$
$$= 2 + 6[81^{1/2} - 1^{1/2}]$$
$$= 2 + 6(9-1)$$
$$= 2 + 6(8)$$
$$V_B(80) = 50 \text{ ft/sec}$$

Rocket A
from table
$V_A(80) = 49$ ft/sec

$50 > 49$,
∴, Rocket B is traveling faster @ $t = 80$ sec

1pt: $6\sqrt{t+1}$
1pt: constant (or definite integral values)
1pt: uses $v(0)$
1pt: $v_B(80)$ + conclusion

OR $v(t) = \int 3(t+1)^{-1/2}\,dt$ $u = t+1$, $du = dt$
$= \int 3u^{-1/2}\,du$
$= 6u^{1/2} + c$
$= 6(t+1)^{1/2} + c$
$v(0) = 6(0+1)^{1/2} + c$
$2 = 6 + c$
$-4 = c$

$v(t) = 6(t+1)^{1/2} - 4$
$v(80) = 6(80+1)^{1/2} - 4$
$= 6(9) - 4$
$= 50$

1pt: correct units in (a) & (b)

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NO CALCULATOR ALLOWED

$\Delta t = 5$ $\Delta t = 15$ $\Delta t = 10$

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

6. A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) Using appropriate units, explain the meaning of $\int_{30}^{60} |v(t)|\, dt$ in terms of the car's motion. Approximate $\int_{30}^{60} |v(t)|\, dt$ using a trapezoidal approximation with the three subintervals determined by the table.

positive #s $\frac{1}{2}(b_1+b_2)h$ not equal!!

$$\int_{30}^{60} |v(t)|\,dt = \frac{1}{2}(14+10)(5) + \frac{1}{2}(10+0)(15) + \frac{1}{2}(0+10)(10)$$

(ft/sec)(sec)

$$= \frac{1}{2}\left[(140)(5) + 10(15) + 10(10)\right]$$

$$= 185 \text{ ft}$$

1pt: trap sum value

$\int_{30}^{60} |v(t)|\,dt$ is total distance in ft that car travels from $t = 30$ sec. to 60 sec.

1pt: explanation

(b) Using appropriate units, explain the meaning of $\int_0^{30} a(t)\, dt$ in terms of the car's motion. Find the exact value of $\int_0^{30} a(t)\, dt$.

velocity

$$\int_0^{30} a(t)\,dt = v(t)\Big|_0^{30}$$

$$= v(30) - v(0)$$

$$= -14 - -20$$

$$= 6 \text{ ft/sec}$$

1pt: value of $\int_0^{30} a(t)\,dt$

$\int_0^{30} a(t)\,dt$ is change in velocity of the car in ft/sec from $t = 0$ to $t = 30$ sec.

1pt: explanation

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Continue problem 6 on page 15

(c) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

find some t on $(0,60)$ where $v(t) > -5$ & $v(t) < -5$ IVT

$v(t)$ cont.

lpt: $v(35) < -5$
$v(50) > -5$

$v(35) = -10$
$v(50) = 0$ } -5 btn -10 & 0

Since $v(35) < -5$ and $v(50) > -5$,

$\exists$ some t on $(0,60)$ where $v(t) = -5$

by IVT.

yes & lpt: refers to IVT in explanation

(d) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

all $a(t)$ values are positive, so not IVT
use MVT, $a(t) = \frac{v(b) - v(a)}{b-a} = 0$

$$a(t) = \frac{v(25) - v(0)}{25 - 0}$$
$$= \frac{-20 - -20}{25} = 0$$

lpt: $v(0) = v(25)$

Since $\frac{v(25) - v(0)}{25 - 0} = 0$ and $a(t)$ cont & $v(t)$ cont,

MVT guarantees there must be a time t on $(0,60)$ when $a(t) = 0$

lpt: refers to MVT

NO CALCULATOR ALLOWED

$\Delta t=2$ $\Delta t=3$ $\Delta t=2$ $\Delta t=4$ $\Delta t=1$

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

5. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $\boxed{0 < t < 12, \text{ the graph of } r \text{ is concave down.}}$ The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$.

(Note: The volume of a sphere of radius r is given by $\boxed{V = \frac{4}{3}\pi r^3}$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

$y - y_1 = m(x - x_1)$

r conc down — over

r conc. up — under

$r - r_1 = r'(5)(t-5)$

$r - 30 = 2.0(t-5)$

$r - 30 = 2.0(5.4-5)$

$r - 30 = 2.0(.4)$

$r - 30 = 0.8$

$r = 30.8$

$r = 30$ @ $t = 5$

$r'(5) = 2.0$ (table)

1pt - estimate

1pt - conc. w/ reason

Estimate is greater than true value b/c r is concave down

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

$\hookrightarrow \frac{dV}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

2pts: $\frac{dV}{dt}$

$\frac{dV}{dt} = 4\pi(30)^2(2)$

1pt - answer

$= 7200\pi$ ft^3/min

$\frac{\text{volume cubic units}}{\text{time}}$

Continue problem 5 on page 13.

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(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

→ not equal

$$\int_0^{12} r'(t)\,dt = 0.5(1) + 0.6(4) + 1.2(2) + 2.0(3) + 4.0(2)$$

$$= .5 + 2.4 + 2.4 + 6 + 8$$

$$= 19.3 \text{ ft}$$

1pt: right approx #

$\int_0^{12} r'(t)\,dt = r(t)\Big|_0^{12}$ $r(12) - r(0)$

$\int_0^{12} r'(t)\,dt$ is change in radius, in ft, from $t = 0$ to 12 minutes

1pt- explanation

1pt: units in (b) + (c)

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

r concave down $\Rightarrow r'' < 0$ $\Rightarrow r'$ dec

r is concave down, so $r'(t)$ dec

1pt: conclusion w/ reason

Approximation in part (c) is less than $\int_0^{12} r'(t)\,dt$ b/c r is concave down, so $r'(t)$ is dec.

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$\Delta t=1$ $\Delta t=2$ $\Delta t=1$

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2. Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

   → need derivative

   (a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

$$L'(5.5) = \frac{L(7)-L(4)}{7-4}$$

$$= \frac{150-126}{3}$$

$$= 8 \text{ people/hr}$$

1pt - estimate
1pt - units

$\frac{1}{2}(b_1+b_2)h$ → not equal.

   (b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

$\frac{1}{b-a}\int_a^b f(x)$

$$\frac{1}{4-0}\int_0^4 L(t)\,dt = \frac{1}{4}\left[\frac{1}{2}(120+156)(1) + \frac{1}{2}(156+176)(2) + \frac{1}{2}(176+126)(1)\right]$$

$$= 155.25 \text{ people}$$

1pt: trapezoid sum

1pt: answer

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Continue problem 2 on page 7.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

L(t) diff'able, so by MVT

on (1,3), $L'(t) = \frac{176-156}{3-1}$

$L'(t) > 0$

L'(t) changes signs once

on (3,4), $L'(t) = \frac{126-176}{4-3}$

$L'(t) < 0$

second time

on (4,7), $L'(t) > 0$

3rd time

on (7,8), $L'(t) < 0$

on (8,9), $L'(t) < 0$

So, by IVT, $L'(t) = 0$ at least 3 times on [0,9]

1pt: considers change in sign of L'(t)

1pt: analysis

1pt: conclusion

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

noon

↳ ∫ rate

rate = derivative

$\int_0^3 r(t)\,dt = 972.784$

$= 973$ tickets

1pt: integrand

1pt: answer

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