Topic 5: Differential Equations

Differential equations are tested every year. The actual solving of the differential equation is usually the main part of the problem, but it is accompanied by a question about its slope field or a tangent line approximation of some sort or something related. BC students may also be asked to approximate using Euler's Method. Large parts of the BC questions are often suitable for AB students and contribute to the AB subscore of the BC exam.

What students should be able to do:

- Find the *general solution* of a differential equation using the method of separation of variables (this is the *only* method tested).
- Find a *particular solution* using the initial condition to evaluate the constant of integration – initial value problem (IVP)
- Understand that proposed solution of a differential equation is a function (not a number) and if it and its derivative are substituted into the given differential equation the resulting equation is true. This may be part of doing the problem even if solving the differential equation is not required (see 2002 BC 5(c))
- Growth-decay problems.
- Draw a slope field by hand.
- Sketch a *particular solution* on a (given) slope field.
- Interpret a slope field.
- For BC only: Use Euler's Method to approximate a solution.
- For BC only: use the method of partial fractions to find the antiderivative after separating the variables.
- For BC only: understand the logistic growth model, its asymptotes, meaning, etc. The exams have never asked students to actually solve a logistic equation IVP.

NO CALCULATOR ALLOWED

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

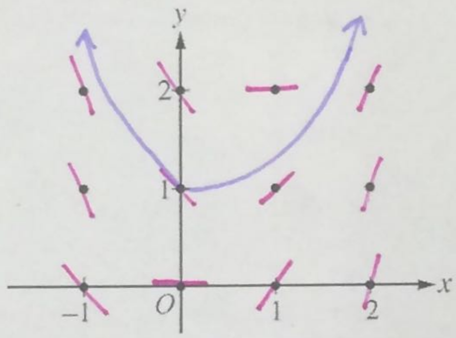
No calculator is allowed for these problems.

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$.

~~(Note: Use the axes provided in the pink test booklet.)~~

$y \backslash x$	-1	0	1	2
2	-4	-2	0	2
1	-3	-1	1	3
0	-2	0	2	4



1pt: zero slopes
1pt: nonzero slopes
1pt: curve through (0,1)

(b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

$dy/dx = 0$

$$\frac{dy}{dx} = 2x - y$$

$$0 = 2(\ln(3/2)) - y$$

$$y = 2\ln\left(\frac{3}{2}\right)$$

1pt: $\frac{dy}{dx} = 0$

1pt: answer

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

y

$\Delta x = \frac{-.4 - 0}{2} = -.2$

	dy/dx	$(\frac{dy}{dx})(\Delta x)$	$\frac{dy}{dx}(\Delta x) + y$
$(0,1)$	-1	$-1(-.2) = .2$	$.2 + 1 = 1.2$
$(-0.2, 1.2)$	-1.6	$-1.6(-.2) = .32$	$.32 + 1.2 = 1.52$

1pt: Euler w/ 2 steps

$f(-.4) \approx 1.52$

1pt: Euler approx to $f(-.4)$

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(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

$\frac{dy}{dx} = 2x - y$

$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$= 2 - (2x - y)$

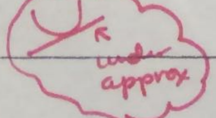
$\frac{d^2y}{dx^2} = 2 - 2x + y$

1pt: $\frac{d^2y}{dx^2}$

$f(-.4)$ in Quad II, pts in Quad II are $x < 0$ & $y > 0$

so $\frac{d^2y}{dx^2} > 0$ in Quad II, ∴ approximation is less than actual $f(-.4)$

1pt: answer w/ reason



NO CALCULATOR ALLOWED

5. Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

$\rightarrow 5x^2 - 6(y-2)^{-1}$

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

$$\left.\frac{dy}{dx}\right|_{(-1,-4)} = 5(-1)^2 - \frac{6}{-4-2}$$
$$= 5 + 1$$
$$= 6$$

$\leftarrow$ ok stop here

1pt: $\left.\frac{dy}{dx}\right|_{(-1,-4)}$

$$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$$

1pt: $\frac{d^2y}{dx^2}$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = 10(-1) + 6(-4-2)^{-2}(6)$$

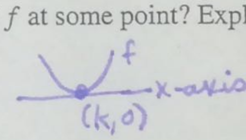
$\leftarrow$ ok Stop here

1pt: $\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)}$

$$= -10 + 36\left(\tfrac{1}{36}\right)$$
$$= -10 + 1 = -9$$

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

$\rightarrow \left.\frac{dy}{dx}\right|_{(k,0)} = 0$



$\frac{dy}{dx} = 0$ @ $(k, 0)$

$5k^2 - \frac{6}{0-2} = 0$

$5k^2 + 3 \neq 0$

$5k^2 + 3 > 0 \quad \forall k$

1pt: $\frac{dy}{dx} = 0$ & $y = 0$

1pt: answer + explanation

So, not possible for x-axis to be tangent to graph of f @ some pt.

Continue problem 5 on page 13.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

$$T_2(x) = f(-1) + f'(-1)(x - -1) + \frac{f''(-1)(x - -1)^2}{2!}$$

$$= -4 + 6(x+1) + \frac{-9}{2}(x+1)^2$$

1pt: quadratic centered @ $x = -1$:
1pt: coefficients

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

$\Delta x = \frac{0 - -1}{2} = \frac{1}{2}$

	$\frac{dy}{dx}$	$\frac{dy}{dx}(\Delta x)$	$\frac{dy}{dx}(\Delta x) + y$
$(-1,-4)$	6	$6(\frac{1}{2}) = 3$	$3 + -4 = -1$
$(-.5,-1)$	$\frac{13}{4}$	$\frac{13}{4}(\frac{1}{2}) = \frac{13}{8}$	$\frac{13}{8} + -1 = \frac{5}{8}$
$(0, 5/8)$			

1pt: Euler's w/ 2 steps

$f(0) \approx \frac{5}{8}$

1pt: Euler approx to $f(0)$

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NO CALCULATOR ALLOWED

5. Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx}$$

$$= 3 + 2(3x + 2y + 1)$$

ok to stop here

$$= 6x + 4y + 5$$

1pt: $3 + 2\frac{dy}{dx}$

1pt: answer

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

$$\frac{dy}{dx} = 3x + 2y + 1 \qquad \frac{dy}{dx} = m + e^{rx}\cdot r$$

if $r \neq 0$

$$3x + 2y + 1 = m + re^{rx}$$

$$3x + 2(mx + b + e^{rx}) + 1 = m + re^{rx}$$

$$3x + 2mx + 2b + 2e^{rx} + 1 = m + re^{rx}$$

$$3 + 2m = 0 \qquad 2e^{rx} = re^{rx} \qquad 2b + 1 = m$$

$$m = -3/2 \qquad 2 = r \qquad 2b + 1 = -3/2$$

$$2b = -5/2$$

$$b = -5/4$$

1pt: $\frac{dy}{dx} = m + e^{rx}\cdot r$

1pt: value of r

1pt: values of m + b

but if $r = 0$, $3x + 2mx + 2b + 2 + 1 = m$

$$3 + 2m = 0 \qquad 2b + 3 = m$$

$$m = -3/2 \qquad 2b + 3 = -3/2$$

$$2b = -9/2$$

$$\boxed{b = -9/4}$$

Continue problem 5 on page 13.

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NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

$\Delta x = 1/2$

	$\frac{dy}{dx}$	$\frac{dy}{dx}\Delta x$	$\frac{dy}{dx}(\Delta x) + y$
$(0,-2)$	-3	$-3(\frac{1}{2}) = -3/2$	$-3/2 + -2 = -7/2$
$(\frac{1}{2}, -\frac{7}{2})$	$-9/2$	$-\frac{9}{2}(\frac{1}{2}) = -9/4$	$-9/4 + -7/2 = -23/4$
$(1, -23/4)$			

1pt: Euler w/ 2 steps

$f(1) \approx -23/4$

1pt: Euler approx answer

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

pt $(0,k)$

$\Delta x = 1$

one step, Euler $\rightarrow$ $y - y_1 = m(\underbrace{x - x_1}_{\Delta x})$

$\left.\frac{dy}{dx}\right|_{(0,k)} = 2k+1$

$y - k = (2k+1)(1)$

$y - k = 2k + 1$

$y = 3k + 1$

1pt: uses $g(0) + g'(0)\cdot 1$

$g(1) \approx 0$ $\rightarrow$

$0 = 3k + 1$

$-1 = 3k$

$-\frac{1}{3} = k$

1pt: value of k

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5. The derivative of a function f is given by $f'(x) = (x - 3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

$\leftarrow f'$ neg to pos

f' pos to neg

f': $-$ on $(0, 3)$, $+$ after 3; test points ① and ④

0 ① 3 ④

f has rel. min @ $x=3$

b/c f' changes from neg to pos @ $x=3$

1pt: rel. min @ $x=3$

1pt: reason

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

$f'<0$ $\quad$ $f''>0$

$f'<0$ on $(0,3)$

$f''(x) = e^x(1) + (x-3)e^x$

$= e^x(1+x-3)$

$0 = e^x(x-2)$

$\cancel{e^x = 0}$ $\quad x-2=0$

$x=2$

2pts: $f''(x)$

f'': $-$ on $(0, 2)$, $+$ after 2; test points ① and ③

0 ① 2 ③

$f''>0$ on $(2,\infty)$

f dec and conc. up on $(2,3)$ b/c

$f'>0$ and $f''>0$ on $(2,3)$

1pt: answer w/ reason

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Continue problem 5 on page 13.

(c) Find the value of $f(3)$.

$$f(x) = f(1) + \int_1^x f'(t)\,dt$$

$$f(3) = 7 + \int_1^3 (x-3)e^x\,dx$$

1pt: use initial condition $f(1) = 7$

	u	dv
$+$	$x-3$	e^x
$-$	1	e^x
$+$	0	e^x

2pts: integration by parts

$$= 7 + \left[(x-3)e^x - e^x\right]\Big|_1^3$$

$$= 7 + (3-3)e^3 - e^3 - \left((1-3)e^1 - e^1\right)$$

1pt: answer

ok stop here

:(

$$= 7 - e^3 + 2e + e$$

$$= 7 - e^3 + 3e$$

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