AP Topic: Power Series (BC only)

Since some graphing calculator can produce Taylor Polynomials, this question appears on the no calculator allowed section. (Questions from 1995 – 1999 before the FR sections was split do not have anything a calculator could do. They are interesting and clever and worth looking at.)

What students should be able to do:

- Find the Taylor (or Maclaurin) polynomial or series for a given function — usually 4 terms and the general term. This may be done by finding the various derivatives, or any other method such as substitution into a known series, long division, the formula for the sum of an infinite geometric series, integration, differentiation, etc.
- Know from memory the Maclaurin series for sin(x), cos(x), e^x, and 1/(1-x).
- Find related series by substitution, differentiation, integration or by adapting one of those above.
- Find the radius of convergence (usually by using the Ratio test, or from a geometric series).
- Find the interval of convergence using the radius and checking the endpoints separately.
- Work with geometric series.
- Use the convergence test separately and when checking the endpoints.
- Find a high-order derivative from the coefficient of a term.
- Estimate the error bound of a Taylor or Maclaurin polynomial by using alternating series error bound or the Lagrange error bound.
- Do not claim that a function is equal to (=) its Taylor or Maclaurin polynomial; it is only approximately equal (≈). This could cost a point.
3. The Taylor series about \( x = 5 \) for a certain function \( f \) converges to \( f(x) \) for all \( x \) in the interval of convergence. The \( n \)th derivative of \( f \) at \( x = 5 \) is given by \( f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n+2)} \), and \( f(5) = \frac{1}{2} \).

(a) Write the third-degree Taylor polynomial for \( f \) about \( x = 5 \).

\[
P_3(x) = f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3
\]

\[
f'(5) = \frac{(x-5)(1)!}{2!(1+2)} = \frac{-1}{2(3)}
\]

\[
f''(5) = \frac{(-1)^2 2!}{2^2 (2+2)} = \frac{2}{4(4)}
\]

\[
f'''(5) = \frac{(-1)^3 3!}{2^3 (3+2)} = \frac{-1}{8(5)}
\]

\[
P_3(x) = \frac{1}{2} + \frac{-1}{6} (x-5) + \frac{2(1) (x-5)^2}{2!} + \frac{-1}{40} (x-5)^3
\]

\[
P_3(x) = \frac{1}{2} - \frac{1}{6} (x-5) + \frac{1}{12} (x-5)^2 - \frac{1}{40} (x-5)^3
\]

3 pts for \( P_3(x) \)
(-1 pt for each wrong term)

(b) Find the radius of convergence of the Taylor series for \( f \) about \( x = 5 \).

\[
f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n! (x-5)^n}{2^n (n+2)} \frac{1}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{2^n (n+2) n!}
\]

\[
\text{Ratio Test:}
\]

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} (x-5)^{n+1}}{2^{n+1} (n+3)} \cdot \frac{2^n (n+2)}{(-1)^n (x-5)^n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (x-5)^{n+1} 2^n (n+2)}{2^{n+1} (n+3) (-1)^n (x-5)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (x-5)^{n+1}}{2^n (n+3)} \cdot \frac{2^{n+1} (n+2)}{(-1)^n (x-5)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{1}{2} (x-5)^3 \right|
\]

\[
\frac{1}{2} (x-5)^3 < 1
\]

\[
|x-5| < 2
\]

Radius of Convergence: 2
(c) Show that the sixth-degree Taylor polynomial for $f$ about $x = 5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

$$|R_6(x)| \leq \frac{\max f^{(7)}(c)}{7!} |x-5|^7$$

Since $f(6)$ is alternating series with $u_n$ decreasing to zero, the error approximating $f(6)$ is less than the 1st omitted term in the series.

$$\max |f^{(7)}(c)| = \frac{|(c-1)^7 \cdot 7!|}{2^7(7+2)}$$

$$= \frac{7!}{2^7(9)}$$

$$|f(6) - P_6(6)| \leq \frac{7!}{2^7(9)} |6-5|^7$$

$$\leq \frac{1}{2^7(9)}$$

$$= \frac{1}{1152}$$

$$\frac{1}{1152} < \frac{1}{1000}$$
6. The Maclaurin series for the function \( f \) is given by
\[
f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots
\]
on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for \( f \). Justify your answer.

\[\lim_{n \to \infty} \left| \frac{(2x)^{n+2}}{(2x)^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(x)^2 \cdot (n+1)}{1 \cdot (2x)^n} \right| = \lim_{n \to \infty} \left| 2x \cdot \left( \frac{n+1}{n+2} \right) \right| = |2x|\]

\[|2x| < 1\]
\[-1 < 2x < 1\]
\[-\frac{1}{2} < x < \frac{1}{2}\]

Test endpoints:
\[x = -\frac{1}{2}\]
\[\sum_{n=0}^{\infty} \frac{(2 \cdot -\frac{1}{2})^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}\]
\[\text{ast*}\]
\[\frac{1}{n+1} > 0\]
\[\frac{1}{h+1} > \frac{1}{h+2}\]
\[\lim_{n \to \infty} \frac{1}{n+1} = 0 \quad \text{(conv. by Alt. Series Test)}\]

\[\therefore \text{ interval of convergence } : \quad -\frac{1}{2} < x < \frac{1}{2}\]
(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

$$f(x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \ldots + \frac{(2x)^{n+1}}{n+1} + \ldots$$

$$f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + \frac{(n+1)(2x)^n - 2}{n+1} + \ldots$$

$$f''(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots$$

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

$$f'(x) = \sum_{n=0}^{\infty} 2(2x)^n$$

geometric series $a = 2$, $r = 2x$

$$= \frac{2}{1 - 2x}$$

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1 - 2\left(-\frac{1}{3}\right)}$$

$$= \frac{2}{1 + \frac{2}{3}}$$

$$= \frac{2}{\frac{5}{3}}$$

$$f'\left(-\frac{1}{3}\right) = \frac{6}{5}$$
3. The Taylor series about \( x = 0 \) for a certain function \( f \) converges to \( f(x) \) for all \( x \) in the interval of convergence. The \( n \)th derivative of \( f \) at \( x = 0 \) is given by
\[
f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2} \quad \text{for} \quad n \geq 2.
\]
The graph of \( f \) has a horizontal tangent line at \( x = 0 \) and \( f(0) = 6 \).

(a) Determine whether \( f \) has a relative maximum, a relative minimum, or neither at \( x = 0 \). Justify your answer.
\[
f'(0) = 0 \quad \Rightarrow \quad \text{critical point at } x = 0
\]
\[
f''(0) = \frac{(-1)^2 \cdot 3!}{5^2 \cdot (2-1)^2} = \frac{-1 \cdot 3}{25}
\]
\[
f''(0) < 0 \quad \Rightarrow \quad 2\text{nd derivative test}
\]
\[
\text{f has rel. max. at } x = 0 \quad \text{b/c } f'(0) = 0 \text{ and } f''(0) < 0
\]

1 pt - answer
1 pt - reason

(b) Write the third-degree Taylor polynomial for \( f \) about \( x = 0 \).
\[
P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3
\]
\[
f(0) = 6 \quad f'(0) = 0 \quad f''(0) = \frac{-1 \cdot 3}{25} \quad f'''(0) = \frac{(-1)^4 \cdot 4!}{5^3(3-1)^2} = \frac{4!}{125 \cdot 4}
\]
\[
P_3(x) = 6 + \frac{-1 \cdot 3}{25} x^2 + \frac{4!}{125 \cdot 4} x^3
\]
\[
P_3(x) = 6 - \frac{3}{25} x^2 + \frac{1}{125} x^3
\]

3 pts - \( P_3(x) \)
(-1 for error in each)
(c) Find the radius of convergence of the Taylor series for $f$ about $x = 0$. Show the work that leads to your answer.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (n+1)!}{5^n (n-1)!^2} x^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (n+1)}{5^n (n-1)^2} x^n$$

1 pt - general term

$\lim_{n \to \infty} \left| \frac{(-1)^{n+2} (n+2) x^{n+1}}{5^{n+1} (n+1)^2} \cdot \frac{5^n (n-1)^2}{(-1)^{n+1} (n+1) x^n} \right|$

1 pt - set up ratio

$= \lim_{n \to \infty} \left| \frac{(-1)(n+2)(n-1)^2 x}{5 n^2 (n+1)} \right|$

1 pt - limit of ratio

$= \left| \frac{-x}{5} \right|$

$= \left| \frac{x}{5} \right|$

$\left| \frac{x}{5} \right| < 1$

$|x| < 5$

1 pt - radius of convergence

Radius of convergence: 5
6. The function \( f \) is defined by \( f(x) = \frac{1}{1 + x^3} \). The Maclaurin series for \( f \) is given by

\[
1 - x^3 + x^6 - x^9 + \ldots + (-1)^n x^{3n} + \ldots,
\]

which converges to \( f(x) \) for \(-1 < x < 1\).

(a) Find the first three nonzero terms and the general term for the Maclaurin series for \( f'(x) \).

\[
f'(x) = -3x^2 + 6x^5 - 9x^8 + \ldots + 3n(-1)^n x^{3n-1} + \ldots
\]

(b) Use your results from part (a) to find the sum of the infinite series \(-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \ldots + (-1)^n \frac{3n}{2^{3n-1}} + \ldots\)

\[
f'(\frac{1}{2}) = -3(\frac{1}{2})^2 + 6(\frac{1}{2})^5 - 9(\frac{1}{2})^8 + \ldots
\]

\[
f(x) = \frac{1}{1 + \frac{x^3}{2}} \]

\[
f'(x) = -3\frac{1}{(1 + x^3/2)^2}(3x^2)
\]

\[
f'(\frac{1}{2}) = -3\frac{1}{(1 + \frac{1/2}{2})^2}(3(\frac{1}{2})^3)
\]

\[
= -\frac{3/4}{(1 + 1/4)^2}
\]

\[
= \frac{-3/4}{8/4} = -\frac{16}{27}
\]
(c) Find the first four nonzero terms and the general term for the Maclaurin series representing \( \int_0^x f(t) \, dt \).

\[
\int_0^x \frac{1}{1+t^3} \, dt = \int_0^x (1 - t^3 + t^6 - t^9 + \cdots + (-1)^n t^{3n} + \cdots) \, dt
\]

\[
= (x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \cdots + (-1)^n \frac{1}{3n+1} x^{3n+1} + \cdots)
\]

1pt - 1st 4 terms
1pt - general term

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate \( \int_0^{1/2} f(t) \, dt \). What are the properties of the terms of the series representing \( \int_0^{1/2} f(t) \, dt \) that guarantee that this approximation is within \( \frac{1}{10,000} \) of the exact value of the integral?

\[
\int_0^{1/2} f(t) \, dt \approx \frac{1}{2} - \frac{1}{4} \left(\frac{1}{2}\right)^4 + \frac{1}{7} \left(\frac{1}{2}\right)^7
\]

Since series in part (c) w/ \( x = \frac{1}{2} \)
has alternating decreasing terms in abs. value + \( \text{Ci}(0) \)
error bounded by abs value of next term.

1pt - prop of terms
1pt - abs value of 4th term
1pt - approx

\[
|\int_0^{1/2} f(t) \, dt - (\frac{1}{2} - \frac{1}{4} \left(\frac{1}{2}\right)^4 + \frac{1}{7} \left(\frac{1}{2}\right)^7)| \leq \frac{1}{10}
\]

\[
= \frac{1}{10240} < \frac{1}{10000}
\]
6. The function \( f \) is defined by the power series
\[
f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots
\]
for all real numbers \( x \) for which the series converges. The function \( g \) is defined by the power series
\[
g(x) = 1 - \frac{x}{21} + \frac{x^2}{41} - \frac{x^3}{61} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots
\]
for all real numbers \( x \) for which the series converges.

(a) Find the interval of convergence of the power series for \( f \). Justify your answer.

\[
f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n nx^n}{n+1}
\]

*Ratio Test*

\[
\lim_{n \to \infty} \frac{(-1)^n (n+1)x^{n+1}}{n+2} \cdot \frac{n}{(-1)^n nx^n}
\]

\[
= \lim_{n \to \infty} \frac{(-1)x \cdot (n+1)(n+1)}{(n+2)n}
\]

\[
= -|x|
\]

\[
\begin{align*}
\quad & |x| < 1 \\
\quad & |x| < 1 \\
\quad & |x| < 1 \\
\quad & |x| < 1
\end{align*}
\]

Check endpoints:

\[
x = -1, \quad \sum_{n=1}^{\infty} \frac{(-1)^n n (-1)^n}{n+1}
\]

\[
= \sum_{n=1}^{\infty} \frac{n}{n+1}
\]

\[
\text{diverges}
\]

\[
\text{a}_n = \frac{n}{n+1} \text{ compares to b}_n = 1 \text{ if } n+1 \text{ diverges,}
\]

Since \( b \) diverges, \( \frac{n}{n+1} \) also diverges

So \( x = -1 \) not included

Interval of convergence: \( -1 < x < 1 \)

Plot - conclusion of both endpoints.

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Continue problem 6 on page 15.
(b) The graph of \( y = f(x) - g(x) \) passes through the point \((0, -1)\). Find \( y'(0) \) and \( y''(0) \). Determine whether \( y \) has a relative minimum, a relative maximum, or neither at \( x = 0 \). Give a reason for your answer.

\[
y'(0) = f'(0) - g'(0) = -\frac{1}{2} - (-\frac{1}{2}) = 0
\]

\[
y''(0) = f''(0) - g''(0) = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}
\]

Since \( y'(0) = 0 \) (\( x \) is a crit #) and \( y''(0) > 0 \) (ycme. up @ \( x = 0 \)),
then \( y \) has rel min @ \( x = 0 \).

STOP

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.
- CHECK TO SEE THAT YOUR AP NUMBER LABEL APPEARS IN THE BOX(ES) ON THE COVER(S).
- MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.
6. Let \( f \) be the function given by \( f(x) = 6e^{-x^3/3} \) for all \( x \).

(a) Find the first four nonzero terms and the general term for the Taylor series for \( f \) about \( x = 0 \).

\[
f(x) = 6 \left( 1 + \frac{-x^3}{6} + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \cdots + \frac{(-x^3)^n}{n!} + \cdots \right)
\]

\[
= 6 \left( 1 - \frac{x^3}{6} + \frac{1}{2 \cdot 3^2} x^6 - \frac{1}{6 \cdot 3^3} x^9 + \cdots + \frac{(-1)^n x^{3n}}{n! 3^n} + \cdots \right)
\]

\[
f(x) = 6 - 2x + \frac{1}{3} x^2 - \frac{1}{27} x^3 + \cdots + \frac{6(-1)^n x^{3n}}{n! 3^n} + \cdots
\]

(b) Let \( g \) be the function given by \( g(x) = \int_0^x f(t) \, dt \). Find the first four nonzero terms and the general term for the Taylor series for \( g \) about \( x = 0 \).

\[
g(x) = \int_0^x f(t) \, dt
\]

\[
= 6x - x^2 + \frac{1}{4} x^3 - \frac{1}{4 \cdot 27} x^4 + \cdots + \frac{6(-1)^n}{n! 3^n} \cdot \frac{1}{n+1} x^{3n+1} + \cdots
\]

\[
= 6x - x^2 + \frac{1}{4} x^3 - \frac{1}{4 \cdot 27} x^4 + \cdots + \frac{6(-1)^n}{(n+1)! 3^n} x^{3n+1} + \cdots
\]
(c) The function \( h(x) = k f'(ax) \) for all \( x \), where \( a \) and \( k \) are constants. The Taylor series for \( h \) about \( x = 0 \) is given by

\[
h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.
\]

Find the values of \( a \) and \( k \).

\[
h(x) = e^x\]

\[
f(x) = 6e^{-\frac{x}{3}}\]

\[
f'(x) = 6e^{-\frac{x}{3}} \cdot -\frac{1}{3} = -2e^{-\frac{x}{3}}\]

\[
h(x) = kf'(ax)\]

\[
e^x = k(-2e^{-ax/3})\]

\[
e^x = -2ke^{-ax/3}\]

\[
1 = -2k \quad \Rightarrow \quad x = -\frac{a}{3}\]

\[
-\frac{1}{2} = k\]

\[
3 = a\]

\[
1 = -\frac{a}{3}\]

\[
-3 = a\]

GO ON TO THE NEXT PAGE.
3. Let $h$ be a function having derivatives of all orders for $x > 0$. Selected values of $h$ and its first four derivatives are indicated in the table above. The function $h$ and these four derivatives are increasing on the interval $1 \leq x \leq 3$.

(a) Write the first-degree Taylor polynomial for $h$ about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

$$P_1(x) = h(2) + h'(2)(x-2)$$

$$P_1(x) = 80 + 128(x-2)$$

$$h(1.9) \approx P_1(1.9) = 67.2$$

$P_1(1.9)$ is less than $h(1.9)$ since $h'$ is increasing on $[1,3]$.

2 pts - $P_1(x)$

1 pt - $P_1(1.9)$

1 pt - $P(1.9) < h(1.9)$ with reason

Continue problem 3 on page 9.
(b) Write the third-degree Taylor polynomial for \( h \) about \( x = 2 \) and use it to approximate \( h(1.9) \).

\[
P_3(x) = h(2) + h'(2)(x-2) + \frac{h''(2)}{2!}(x-2)^2 + \frac{h'''(2)}{3!}(x-2)^3
\]

\[
P_3(x) = 80 + 128(x-2) + \frac{488}{2!}(x-2)^2 + \frac{488/3}{3!}(x-2)^3
\]

\[
P_3(x) = 80 + 128(x-2) + \frac{488}{2!}(x-2)^2 + \frac{488/3}{3!}(x-2)^3
\]

\[
2pt - P_3(x)
\]

\[
h(1.9) \approx P_3(1.9) = 67.988
\]

\[
1pt - P_3(1.9)
\]

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for \( h \) about \( x = 2 \) approximates \( h(1.9) \) with error less than \( 3 \times 10^{-4} \).

\[
h''(x) \text{ is inc on } [1.3],
\]

so max on \([1.9, 2]\) is \( \max |h''(c)| \leq \frac{584}{9}\)

\[
|h(1.9) - P_3(1.9)| \leq \frac{\max |h''(c)|}{4!} |x-2|^4
\]

\[
\leq \frac{584/9}{4!} |1.9-2|^4
\]

\[
= 2.704 \times 10^{-4}
\]

which is \( \leq 3 \times 10^{-4} \)

\[
1pt - \text{Lagrange error estimate}
\]

\[
1pt - \text{reasoning}
\]

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.
6. Consider the logistic differential equation \( \frac{dy}{dt} = \frac{y}{8}(6 - y) \). Let \( y = f(t) \) be the particular solution to the differential equation with \( f(0) = 8 \).

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points \((3, 2)\) and \((0, 8)\).

(b) Use Euler's method, starting at \( t = 0 \) with two steps of equal size, to approximate \( f(1) \).

\[
\Delta x = \frac{b-a}{n} = \frac{1-0}{2} = \frac{1}{2}
\]

\[
\begin{array}{c|c|c|c}
(0, 8) & \frac{1}{2} & \frac{dy}{dt} & \frac{dy}{dt} \Delta x \\
\hline
 & & \Delta x & \Delta t = \frac{1-0}{2} = \frac{1}{2} \\
\hline
(0, 8) & \frac{1}{2} & -2 & -2(\frac{1}{2}) = -1 \\
(0.5, 7) & \frac{1}{2} & -\frac{7}{8} & -\frac{7}{8}(\frac{1}{2}) = -\frac{7}{16} \\
(1, 7.5) & \frac{1}{2} & -7 & -7 + 7 = 0 \\
\end{array}
\]

\( f(1) \approx 7 - \frac{7}{16} \)
(c) Write the second-degree Taylor polynomial for \( f \) about \( t = 0 \), and use it to approximate \( f(1) \).

\[ P_2(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 \]

\[ f(0) = 8 \]
\[ f'(0) = \frac{du}{dt} \bigg|_{t=0} = \frac{8}{8} (6-8) = -2 \]
\[ f''(t) = (6-4)(\frac{1}{8} \frac{du}{dt}) + \frac{u}{8} (-1) \frac{du}{dt} \]
\[ f''(0) = (6-4)(\frac{1}{8} \frac{8}{8}) + \frac{8}{8} (-1) -2 \]
\[ = \frac{3}{2} + 2 \]
\[ = \frac{7}{2} \]

\[ P_2(t) = 8 - 2t + \frac{7}{2}t^2 \]

\[ f(1) \approx P_2(1) = 8 - 2(1) + \frac{7}{2}(1)^2 \]

\[ f(1) \approx 29/4 \]

2 pts - \( f''(t) \) or \( \frac{d^2y}{dt^2} \)

1 pt - \( P_2(t) \)

1 pt - \( f(1) \)

(d) What is the range of \( f \) for \( t \geq 0 \)?

Range for \( t \geq 0 \), \((6, \infty]\)
6. Let $f$ be the function given by $f(x) = \frac{2x}{1 + x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $f$ about $x = 0$.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$
$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots + (-x^2)^n + \cdots$$
$$\frac{1}{1+x^2} = 1 = x^2 + x^4 - x^6 + \cdots + (-x^2)^n + \cdots$$
$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + 2x(-x^2)^n + \cdots$$

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$
$$f(1) = \sum_{n=0}^{\infty} (-1)^n 2(1)^{2n+1}$$
$$= \sum_{n=0}^{\infty} 2(-1)^n$$

* $n^{th}$ term test *

$$\lim_{n \to \infty} 2(-1)^n \neq 0 \therefore \text{series diverges}$$

**Series does NOT converge** when evaluated at $x = 1$
(c) The derivative of \( \ln(1 + x^2) \) is \( \frac{2x}{1 + x^2} \). Write the first four nonzero terms of the Taylor series for 
\[ \ln(1 + x^2) \] about \( x = 0 \).

\[
\begin{align*}
\frac{d}{dx} \ln(1 + x^2) &= \int \frac{2x}{1 + x^2} \\
\ln(1 + x^2) &= \int \frac{2t}{1 + t^2} dt \\
&= \int (2t - 2t^3 + 2t^5 - 2t^7 + \ldots) dt \\
&= \left( t^2 - \frac{1}{3} t^4 + \frac{1}{5} t^6 - \frac{1}{7} t^8 + \ldots \right) \bigg|_0^x \\
&= x^2 - \frac{1}{3} x^4 + \frac{1}{5} x^6 - \frac{1}{7} x^8 + \ldots
\end{align*}
\]

(d) Use the series found in part (c) to find a rational number \( A \) such that \( |A - \ln \left( \frac{5}{4} \right)| < \frac{1}{100} \). Justify your answer.

\[
\begin{align*}
\ln \left( \frac{5}{4} \right) &= \ln (1 + \frac{1}{4}) \\
&= \ln \left( 1 + \left( \frac{1}{2} \right)^2 \right) \\
&= \left( \frac{1}{2} \right)^2 - \frac{1}{2} \left( \frac{1}{2} \right)^4 + \frac{1}{3} \left( \frac{1}{2} \right)^6 - \frac{1}{4} \left( \frac{1}{2} \right)^8 + \ldots \\
&= \frac{1}{4} - \frac{1}{16} + \frac{1}{48} - \frac{1}{128} + \ldots \\
&= \text{max of remainder}
\end{align*}
\]

\[
\begin{align*}
|A - \ln \left( \frac{5}{4} \right)| &\leq \frac{1}{5} \left( \frac{1}{2} \right)^6 \\
&= \frac{1}{5} \left( \frac{1}{64} \right) \\
&= \frac{1}{192} < \frac{1}{100}
\end{align*}
\]