AP Topic: Power Series (BC only)

Since some graphing calculator can produce Taylor Polynomials, this question appears on the no calculator allowed section. (Questions from 1995 – 1999 before the FR sections was split do not have anything a calculator *could* do. They are interesting and clever and worth looking at.)

What students should be able to do:

- Find the Taylor (or Maclaurin) polynomial or series for a given function – usually 4 terms and the general term). This may be done by finding the various derivatives, or any other method such as substitution into a known series, long division, the formula for the sum of an infinite geometric series, integration, differentiation, etc.
- Know from memory the Maclaurin series for $\sin(x)$, $\cos(x)$, e^x, and $\frac{1}{1-x}$.
- Find related series by substitution, differentiation, integration or by adapting one of those above.
- Find the radius of convergence (usually by using the Ratio test, or from a geometric series).
- Find the interval of convergence using the radius and checking the endpoints separately.
- Work with geometric series.
- Use the convergence test separately and when checking the endpoints.
- Find a high-order derivative from the coefficient of a term.
- Estimate the error bound of a Taylor or Maclaurin polynomial by using *alternating series error bound* or the *Lagrange error bound.*
- *Do not* claim that a function is equal to (=) its Taylor or Maclaurin polynomial; it is only approximately equal ($\approx$). This could cost a point.

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3. The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$, and $\boxed{f(5) = \frac{1}{2}}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

$$P_3(x) = f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3$$

$$f'(5) = \frac{(-1)^1(1)!}{2^1(1+2)} = \frac{-1}{2(3)} \qquad f''(5) = \frac{(-1)^2 \cdot 2!}{2^2(2+2)} = \frac{2}{4(4)} \qquad f'''(5) = \frac{(-1)^3(3!)}{2^3(3+2)} = \frac{-1 \cdot 3!}{8(5)}$$

$$P_3(x) = \frac{1}{2} + \frac{-1}{6}(x-5) + \frac{2/16}{2!}(x-5)^2 + \frac{\frac{-1 \cdot 3!}{40}}{3!}(x-5)^3$$

$$\boxed{P_3(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3}$$

3pts for $P_3(x)$ (-1 for each wrong term)

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

$$f(x) = \sum_{n=0}^{\infty} \frac{\frac{(-1)^n n!}{2^n(n+2)}}{n!}(x-5)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+2)}(x-5)^n$$

1 pt - general term

1 pt - setup ratio

1 pt - limit ratio

* Ratio test *

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)} \cdot \frac{2^n(n+2)}{(-1)^n(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}(-1)^1\cancel{(x-5)^n}(x-5)^1 \cdot \cancel{2^n}(n+2)}{\cancel{2^n} \cdot 2^1(n+3)\cancel{(-1)^n}\cancel{(x-5)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{2}(x-5)\frac{(n+2)}{(n+3)}\right|$$

$$= \left|\frac{1}{2}(x-5)\right|$$

$$\frac{1}{2}|x-5| < 1$$

$$|x-5| < 2$$

1 pt - gets radius of convergence

$\boxed{\text{Radius of Convergence} : 2}$

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

$$|R_6(x)| \le \frac{|\max f^{(7)}(c)|}{7!}|x-5|^7$$

since f(6) is alternating series w/ u_n decreasing to zero,
error approximating f(6) is less than the 1st omitted term in the series

$$\max|f^{(7)}(c)| = \left|\frac{(-1)^7 \cdot 7!}{2^7(7+2)}\right| = \frac{7!}{2^7(9)}$$

$$|f(6) - P_6(6)| \le \frac{\frac{7!}{2^7(9)}}{7!}|6-5|^7$$

$$\le \frac{1}{2^7(9)}$$

$$= \frac{1}{1152}$$

$$\frac{1}{1152} < \frac{1}{1000}$$

1 pt - shows error bound $< \frac{1}{1000}$

1 pt - refers to alt. series + error bound found in next term

NO CALCULATOR ALLOWED

6. The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

* Ratio*

$$\lim_{n\to\infty} \left| \frac{(2x)^{n+2}}{n+2} \cdot \frac{n+1}{(2x)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2x)^n \cdot (2x)^2 (n+1)}{(n+2)(2x)^n (2x)^1} \right|$$

$$= \lim_{n\to\infty} \left| 2x \cdot \frac{(n+1)}{n+2} \right|$$

$$= |2x|$$

1pt - setup ratio
1pt - limit of ratio

$$|2x| < 1$$
$$-1 < 2x < 1$$
$$-\tfrac{1}{2} < x < \tfrac{1}{2}$$

1pt - interval

test endpts:

$x = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{(2\cdot\frac{1}{2})^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

LCT

$a_n = \frac{1}{n+1}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges

1pt - correctly checks right endpt

$x = -\frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{(2\cdot -\frac{1}{2})^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

AST

$$\left.\begin{array}{l} \frac{1}{n+1} > 0 \checkmark \\ \frac{1}{n+1} > \frac{1}{n+2} \checkmark \\ \lim_{n\to\infty} \frac{1}{n+1} = 0 \checkmark \end{array}\right\}$$ so, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges by Alt. Series Test

1pt - correctly checks left endpt

$\boxed{\therefore\text{, interval of convergence}: -\tfrac{1}{2} \le x < \tfrac{1}{2}}$

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NO CALCULATOR ALLOWED

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

$$f(x) = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

$$f'(x) = 2 + 4x + 8x^2 + 16x^3 + \cdots + \frac{(n+1)(2x)^n \cdot 2}{n+1} + \cdots$$

$$\boxed{f'(x) = 2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots}$$

1pt - 1st 4 terms

1pt - general term

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

$$f'(x) = \sum_{n=0}^{\infty} 2(2x)^n$$

geometric series $a = 2,\ r = 2x$

$$= \frac{2}{1-2x}$$

$$f'(-\tfrac{1}{3}) = \frac{2}{1-2(-\frac{1}{3})}$$

$$= \frac{2}{1+2/3}$$

$$= \frac{2}{5/3}$$

$$\boxed{f'(-\tfrac{1}{3}) = \tfrac{6}{5}}$$

1pt - sub in $x = -\frac{1}{3}$ into infinite series or expresses series in closed form

1pt - answer

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3. The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2} \text{ for } n \geq 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

$f'(0) = 0 \rightarrow$ crit # @ $x = 0$

$f''(0) = \frac{(-1)^3 (3)!}{5^2 (2-1)^2}$

$= \frac{-1 \cdot 3!}{25}$

$f''(0) < 0$ 2nd derivative test ... :)

$\boxed{f \text{ has rel. max @ } x=0 \text{ b/c } f'(0)=0 \text{ and } f''(0)<0}$

1pt - answer

1pt - reason

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$

$f(0) = 6$ $\quad f'(0) = 0$ $\quad f''(0) = \frac{-1 \cdot 3!}{25}$ $\quad f'''(0) = \frac{(-1)^4 (4)!}{5^3 (3-1)^2} = \frac{4!}{125 \cdot 4}$

$P_3(x) = 6 + \frac{\frac{-1 \cdot 3!}{25}}{2!} x^2 + \frac{\frac{4!}{125 \cdot 4}}{3!} x^3$

$\boxed{P_3(x) = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3}$

3pts - $P_3(x)$

(-1 for each error)

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

$$f(x) = \sum_{n=0}^{\infty} \frac{\frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2}}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)}{5^n(n-1)^2} \cdot x^n$$

1pt - general term

* Ratio *

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+2)}{5^{n+1}(n)^2} \cdot x^{n+1} \cdot \frac{5^n(n-1)^2}{(-1)^{n+1}(n+1)x^n} \right|$$

1pt - setup ratio

$$= \lim_{n\to\infty} \left| \frac{(-1)(n+2)(n-1)^2 x}{5n^2(n+1)} \right|$$

1pt - limit of ratio

$$= \left| \frac{-x}{5} \right|$$

$$= \left| \frac{x}{5} \right|$$

$$\left| \frac{x}{5} \right| < 1$$

$$|x| < 5$$

1pt - radius of convergence

$\boxed{\text{Radius of convergence : } 5}$

NO CALCULATOR ALLOWED

6. The function f is defined by $f(x) = \frac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

$f'(x) = -3x^2 + 6x^5 - 9x^8 + \ldots + 3n(-1)^n x^{3n-1} + \ldots$

1pt – 1st 3 terms
1pt – general term

(b) Use your results from part (a) to find the sum of the infinite series $-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots$.

$f'(\frac{1}{2}) = -3(\frac{1}{2})^2 + 6(\frac{1}{2})^5 - 9(\frac{1}{2})^8 + \cdots$

$f(x) = \frac{1}{1+x^3} = (1+x^3)^{-1}$

$f'(x) = -1(1+x^3)^{-2}(3x^2)$

$f'(\frac{1}{2}) = -1(1+(\frac{1}{2})^3)^{-2}(3(\frac{1}{2})^2)$

$= \frac{-3/4}{(1+1/8)^2}$

$= \frac{-3/4}{81/64} = -\frac{16}{27}$

1pt - $f'(x)$
1pt - $f'(\frac{1}{2})$

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Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

$$\int_0^x \frac{1}{1+t^3}\,dt = \int_0^x \left(1 - t^3 + t^6 - t^9 + \cdots + (-1)^n t^{3n} + \cdots\right) dt$$

$$= \left(t - \frac{1}{4}t^4 + \frac{1}{7}t^7 - \frac{1}{10}t^{10} + \cdots + (-1)^n \frac{1}{3n+1} t^{3n+1} + \cdots\right)\Big|_0^x$$

$$= x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \cdots (-1)^n \frac{x^{3n+1}}{3n+1} + \cdots$$

1pt – 1st 4 terms
1pt – general term

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\frac{1}{10{,}000}$ of the exact value of the integral?

$$\int_0^{1/2} f(t)\,dt \approx \frac{1}{2} - \frac{1}{4}\left(\frac{1}{2}\right)^4 + \frac{1}{7}\left(\frac{1}{2}\right)^7$$

1pt - approximation

Since series in part (c) w/ $x=\frac{1}{2}$ has alternating decreasing terms in abs. value + limit 0, error bounded by abs value of next term.

1pt - property of terms

$$\Bigg| \underbrace{\int_0^{1/2} f(t)\,dt}_{\text{@}x=1/2 \text{ function}} - \underbrace{\left(\frac{1}{2} - \frac{1}{4}\left(\frac{1}{2}\right)^4 + \frac{1}{7}\left(\frac{1}{2}\right)^7\right)}_{\text{poly approx @ }x=1/2} \Bigg| \le \frac{\left(\frac{1}{2}\right)^{10}}{10}$$

$$= \frac{1}{10240} < \frac{1}{10000}$$

1pt - abs value of 4th term $< \frac{1}{10000}$

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6. The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{n+1}$

Ratio Test

$\lim_{n\to\infty} \frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n n x^n}$

$= \lim_{n\to\infty} (-1)x \cdot \frac{(n+1)(n+1)}{(n+2)n}$

$= -1x$

$|-1x| < 1$

$|-1||x| < 1$

$|x| < 1$

$-1 < x < 1$

check endpts:

$x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n n(-1)^n}{n+1}$

$= \sum_{n=1}^{\infty} \frac{n}{n+1}$

diverges

$a_n = \frac{n}{n+1}$ compares to $b_n = 1$

since $\sum 1$ diverges,

$\sum_{n=1}^{\infty} \frac{n}{n+1}$ also diverges

So $x=-1$, not included

$x=1$, $\sum_{n=1}^{\infty} \frac{(-1)^n n(1)^n}{n+1}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

AST

① $\frac{n}{n+1} > 0$ for $1<n<\infty$

② $\frac{n}{n+1} > \frac{n+1}{n+2}$

③ $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$;

So series diverges,

∴, $x=1$ not included

$\boxed{\text{interval of convergence: } -1 < x < 1}$

1pt - set up ratio

1pt - limit of ratio

1pt - radius of convergence

1pt - consider both endpts

1pt - conclusion of both endpts.

Continue problem 6 on page 15.

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(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

$y(x) = f(x) - g(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots \quad - \left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right)$

$y'(x) = f'(x) - g'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots \quad - \left(-\frac{1}{2} + \frac{2x}{4!} - \frac{3x^2}{6!} + \cdots\right)$

$y'(0) = f'(0) - g'(0) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$

$\boxed{y'(0) = 0}$

1pt - $y'(0)$

$y''(x) = f''(x) - g''(x) = \frac{4}{3} - \frac{18}{4}x + \cdots \quad - \left(\frac{2}{4!} - \frac{6x}{6!} + \cdots\right)$

$y''(0) = f''(0) - g''(0) = \frac{4}{3} - \frac{2}{4!}$

$y''(0) = \frac{4}{3} - \frac{1}{12}$

$\boxed{y''(0) = \frac{15}{12} \text{ or } \frac{5}{4}}$

1pt - $y''(0)$

Since $y'(0) = 0$ (x is a crit #) and $y''(0) > 0$ (y conc. up @ $x = 0$), then y has rel. min @ $x = 0$.

1pt - conclusion,
1pt - reason

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STOP

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.
- CHECK TO SEE THAT YOUR AP NUMBER LABEL APPEARS IN THE BOX(ES) ON THE COVER(S).
- MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.

6. Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

$$f(x) = 6\left(1 + \left(-\tfrac{x}{3}\right) + \frac{(-x/3)^2}{2!} + \frac{(-x/3)^3}{3!} + \cdots + \frac{(-x/3)^n}{n!} + \cdots\right)$$

$$= 6\left(1 - \tfrac{1}{3}x + \frac{1}{2\cdot 3^2}x^2 - \frac{1}{6\cdot 3^3}x^3 + \cdots + \frac{(-1)^n x^n}{n!\,3^n} + \cdots\right)$$

$$f(x) = 6 - 2x + \tfrac{1}{3}x^2 - \tfrac{1}{27}x^3 + \cdots + \frac{6(-1)^n x^n}{n!\,3^n} + \cdots$$

1pt - 2 of 4 terms correct
1pt - remaining terms
1pt - general term

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

~~$g(x) = 6(x$~~

$$g(x) = \int_0^x f(t)\,dt$$

$$= 6x - x^2 + \tfrac{1}{9}x^3 - \frac{1}{4\cdot 27}x^4 + \cdots + \frac{6(-1)^n}{n!\,3^n}\cdot\frac{1}{n+1}x^{n+1} + \cdots$$

$$= 6x - x^2 + \tfrac{1}{9}x^3 - \frac{1}{4\cdot 27}x^4 + \cdots + \frac{6(-1)^n}{(n+1)!\,3^n}x^{n+1} + \cdots$$

Continue problem 6 on page 15.

(c) The function h satisfies $h(x) = kf'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

$h(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

$= e^x$

$f(x) = 6e^{-x/3}$

$f'(x) = 6e^{-x/3} \cdot -\frac{1}{3}$

$= -2e^{-x/3}$

$h(x) = kf'(ax)$

$e^x = k(-2e^{-ax/3})$

$e^x = -2ke^{-ax/3}$

$1 = -2k$ $\qquad$ $x = -ax/3$

$\boxed{-\frac{1}{2} = k}$ $\qquad$ $1 = -a/3$

$\boxed{-3 = a}$

1pt - computes $kf'(ax)$

1pt - recognizes $h(x) = e^x$ or equates 2 series for $h(x)$

1pt - values for a & k

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GO ON TO THE NEXT PAGE.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

3. Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

$P_1(x) = h(2) + h'(2)(x-2)$

$\boxed{P_1(x) = 80 + 128(x-2)}$ 2pts - $P_1(x)$

$\boxed{h(1.9) \approx P_1(1.9) = 67.2}$ 1pt - $P_1(1.9)$

$P_1(1.9)$ is less than $h(1.9)$ b/c h' is increasing on $[1,3]$

1pt - $P(1.9) < h(1.9)$ w/ reason

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Continue problem 3 on page 9.

(b) Write the third-degree Taylor polynomial for h about $\boxed{x = 2}$ and use it to approximate $h(1.9)$.

$P_3(x) = h(2) + h'(2)(x-2) + \frac{h''(2)}{2!}(x-2)^2 + \frac{h'''(2)}{3!}(x-2)^3$

$P_3(x) = 80 + 128(x-2) + \frac{488/3}{2!}(x-2)^2 + \frac{448/3}{3!}(x-2)^3$

$\boxed{P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3}$ 2pts - $P_3(x)$

$\boxed{h(1.9) \approx P_3(1.9) = 67.988}$

1pt - $P_3(1.9)$

$\max \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1}$

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}. $\rightarrow n = 3$, so max of $h^4(c)$

$h^4(x)$ is inc on $[1, 3]$,
so max on $[1.9, 2]$ is $\max|h^4(2)| \le \frac{584}{9}$

$|h(1.9) - P_3(1.9)| \le \frac{\max|h^4(c)|}{4!} |x-2|^4$

$\le \frac{584/9}{4!} |1.9-2|^4$ 1pt - Lagrange error estimate

$= 2.704 \times 10^{-4}$

which is $< 3 \times 10^{-4}$ 1pt - reasoning

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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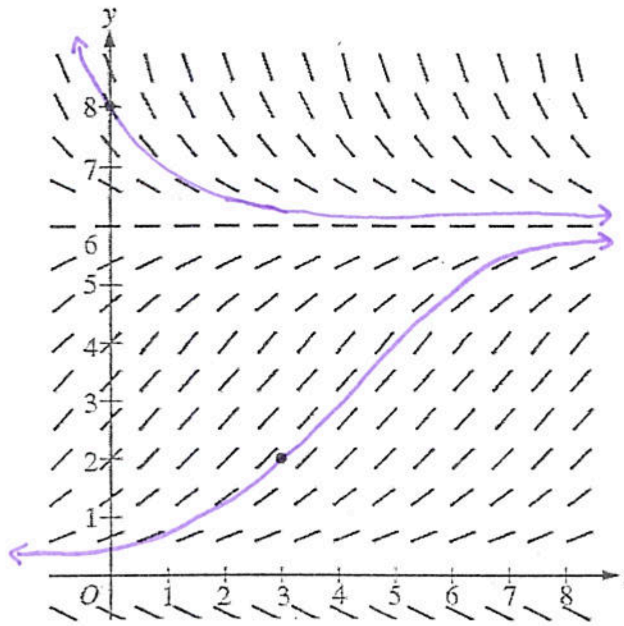
NO CALCULATOR ALLOWED

6. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.



1pt - solution through (0,8)

1pt - solution thru (3,2)

(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

$\hookrightarrow \Delta x = \frac{b-a}{n} = \frac{1-0}{2} = \frac{1}{2}$

	Δx	dy/dx	$dy/dx\,\Delta x$	$dy/dx\,\Delta x + y$
$(0,8)$	$\frac{1}{2}$	-2	$-2(\frac{1}{2}) = -1$	$-1 + 8 = 7$
$(0.5,7)$	$\frac{1}{2}$	$-7/8$	$-\frac{7}{8}(\frac{1}{2}) = \frac{-7}{16}$	$7 + \frac{-7}{16}$
$(1, 7-\frac{7}{16})$				

1pt - Euler's method w/ 2 steps

$\boxed{f(1) \approx 7 - \frac{7}{16}}$

1pt - approx of f(1)

Do not write beyond this border.

Continue problem 6 on page 15.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

$P_2(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2$

$f(0) = 8$

$f'(0) = \left.\frac{dy}{dt}\right|_{t=0}$

$= \frac{8}{8}(6-8) = -2$

$f''(t) = (6-y)\left(\frac{1}{8}\frac{dy}{dt}\right) + \frac{y}{8}\left(-1\cdot\frac{dy}{dt}\right)$

$f''(0) = (6-8)\left(\frac{1}{8}\cdot -2\right) + \frac{8}{8}(-1\cdot -2)$

$= (-2)(-\frac{1}{4}) + 2$

$= \frac{1}{2} + 2$

$= \frac{5}{2}$

2pts - $f''(t)$ or d^2y/dt^2

$P_2(t) = 8 + -2t + \frac{5/2}{2!}t^2$

$f(1) \approx P_2(1) = 8 - 2(1) + \frac{5}{4}(1)^2$

$\boxed{f(1) \approx 29/4}$

1pt - $P_2(t)$

1pt - $f(1)$

(d) What is the range of f for $t \geq 0$?

see slope field

Since using $f(0) = 8$

1pt - answer

$\boxed{\text{range for } t \geq 0,\ (6, 8]}$

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NO CALCULATOR ALLOWED

6. Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots + (-x^2)^n + \cdots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-x^2)^n + \cdots$$

$$\frac{2x}{1+x^2} = \boxed{2x - 2x^3 + 2x^5 - 2x^7 + \cdots + \underbrace{2x(-x^2)^n}_{\text{OR } (-1)^n \cdot 2x \cdot x^{2n}} + \cdots}$$

$(-1)^n \cdot 2x^{2n+1}$

1pt - 2 of 1st 4 terms

1pt - remaining terms

1pt - general term

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

$$f(1) = \sum_{n=0}^{\infty} (-1)^n 2(1)^{2n+1}$$

$$= \sum_{n=0}^{\infty} 2 \cdot (-1)^n$$

* n^{th} term test *

$\lim_{n\to\infty} 2(-1)^n \neq 0$, $\therefore$ series diverges

1pt - answer w/reason

$\boxed{\text{Series does NOT converge to } f(1) \text{ when evaluated @ } x=1}$

Continue problem 6 on page 15.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

$$\int \frac{d}{dx}\left(\ln(1+x^2)\right) = \int \frac{2x}{1+x^2}$$

$$\ln(1+x^2) = \int_0^x \frac{2t}{1+t^2}\,dt$$

$$= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$$

$$= \left(t^2 - \tfrac{1}{2}t^4 + \tfrac{1}{3}t^6 - \tfrac{1}{4}t^8 + \cdots\right)\Big|_0^x$$

$$\boxed{\ln(1+x^2) = x^2 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^6 - \tfrac{1}{4}x^8 + \cdots}$$

1pt - 2 of 1st 4 terms

1pt - remaining terms

(d) Use the series found in part (c) to find a rational number A such that $\underbrace{\left|A - \ln\left(\frac{5}{4}\right)\right|}_{\text{remainder}} < \frac{1}{100}$. Justify your answer.

$$\ln(5/4) = \ln\left(1+\tfrac{1}{4}\right)$$

$$= \ln\left(1+\left(\tfrac{1}{2}\right)^2\right)$$

$$= \underbrace{\left(\tfrac{1}{2}\right)^2 + -\tfrac{1}{2}\left(\tfrac{1}{2}\right)^4}_{\text{let } A = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4} + \underbrace{\tfrac{1}{3}\left(\tfrac{1}{2}\right)^6 - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^8 + \cdots}_{\text{remainder}}$$

1pt - uses $x = \frac{1}{2}$

1pt - value of A

1pt - reason

max of remainder

$$\left|A - \ln\left(\tfrac{5}{4}\right)\right| \le \tfrac{1}{3}\left(\tfrac{1}{2}\right)^6$$

$$= \tfrac{1}{3}\left(\tfrac{1}{64}\right)$$

$$= \frac{1}{192} < \frac{1}{100}$$

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