NO CALCULATOR ALLOWED

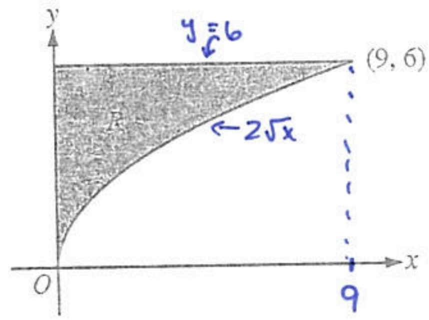
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



4. Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R. → top - bottom ... ☺

$$\text{Area of } R = \int_0^9 (6 - 2\sqrt{x})\,dx$$

1pt - integrand

$$= \int_0^9 (6 - 2x^{1/2})\,dx$$

$$= 6x - 2\left(\tfrac{2}{3}x^{3/2}\right)\Big|_0^9$$

1pt - antiderivative

$$= 6(9) - \tfrac{4}{3}(9)^{3/2} - (0)$$

1pt - answer

ok to stop here

$$= 54 - \tfrac{4}{3}(\sqrt{9})^3$$

$$= 54 - \tfrac{4}{3}\cdot 27$$

$$= 54 - 36$$

$$= 18$$

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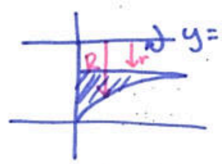
Continue problem 4 on page 11.

(b) Write, but $\boxed{\text{do not evaluate}}$, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.



* WASHER *

outside radius

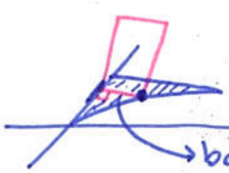
inside radius

$$V = \pi \int_0^9 \left[(7 - 2\sqrt{x})^2 - (7-6)^2 \right] dx$$

2pts - integrand
1pt - limits + constant (π)

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a $\boxed{\text{rectangle}}$ whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

$\to A = bh$ $\to h = 3\,\text{base}$



$y = 2\sqrt{x}$

$\frac{y}{2} = \sqrt{x} \to x = \left(\frac{y}{2}\right)^2$

$\to \text{base} = \frac{y^2}{4} - 0$

$A = bh$

$A = b(3b)$

$A = \left(\frac{y^2}{4}\right)\left(3 \cdot \frac{y^2}{4}\right)$

$A = 3\left(\frac{y^2}{4}\right)^2$

y #s $\to$

$$V = \int_0^6 \left(3\left(\frac{y^2}{4}\right)^2 \right) dy$$

2pt - integrand
1pt - answer

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NO CALCULATOR ALLOWED

5. Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

   (a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

a b

$\Delta x = \frac{0-1}{2} = -\frac{1}{2}$

$1 - 0 = 1$
$1 - -\frac{1}{2} = 3/2$

	Δx	$\frac{dy}{dx}$	$\Delta x \frac{dy}{dx}$	$y + \Delta x \frac{dy}{dx}$
$(1, 0)$	$-1/2$	1	$-1/2$	$0 + -1/2 = -1/2$
$(1/2, -1/2)$	$-1/2$	$3/2$	$-3/4$	$-1/2 + -3/4 = -5/4$
$(0, -5/4)$				

$-\frac{1}{2} + -\frac{3}{4}$
$-\frac{2}{4} + -\frac{3}{4}$

1pt – Euler w/ 2 steps

$f(0) = -5/4$

1pt – answer

   (b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

$$\frac{\lim_{x \to 1} f(x)}{\lim_{x \to 1} (x^3 - 1)} = \frac{f(1)}{1^3 - 1} = \frac{0}{0} \quad \text{L'Hôpital}$$

1pt – use of L'Hôpital

$$\lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\left.\frac{dy}{dx}\right|_{x=1}}{3(1)^2} = \frac{1}{3}$$

1pt – answer

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Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

S y's w/ dy's + x's w/ dx's

$\frac{dy}{dx} = 1 - y$

$dy = (1-y)\,dx$

$\int \frac{1}{1-y}\,dy = \int dx$

1pt - separate variables

$-\ln|1-y| = x + C$

$-\ln|1-0| = 1 + C$

$0 = 1 + C$

$-1 = C$

1pt - antiderivative

1pt - "+C"

1pt - initial condition

$-\ln|1-y| = x - 1$

$\ln|1-y| = -x + 1$

$|1-y| = e^{-x+1}$

$1-y = \pm e^{-x+1}$

$1-y = e^{-x+1}$

$-y = e^{-x+1} - 1$

$y = -e^{-x+1} + 1$

1pt - solves for y

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$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

6. The function f, defined above, has derivatives of all orders. Let g be the function defined by

$g(x) = 1 + \int_0^x f(t)\,dt.$

(a) Write the first three nonzero terms and the general term of the Taylor series for cos x about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

1pt – cos x

$$\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$\frac{\cos x - 1}{x^2} = \frac{-x^2/2!}{x^2} + \frac{x^4/4!}{x^2} - \frac{x^6/6!}{x^2} + \cdots + \frac{\frac{(-1)^n x^{2n}}{(2n)!}}{x^2} + \cdots$$

1pt – 3 terms
1pt – general term

$$= -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + \frac{(-1)^n x^{2n-2}}{(2n)!} + \cdots$$

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

$f''(x) < 0$ @ crit #

$f''(x) > 0$ @ crit #

$f''(x) = 0$ @ crit #

$\hookrightarrow f'(x) = 0$ crit # @ $x = 0$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$$f(x) = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots$$

$f'(0) = 0$ no x-term

so $x = 0$ crit #.

1pt – finds $f'(0) = 0$

$f''(0) = \frac{1}{4!}$ coeff. of x^2 term

1pt – answer w/ reason

Since $f''(0) > 0$, then $x = 0$ is rel. max

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Continue problem 6 on page 15.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

$\to P_5(x)$

$g(x) = 1 + \int_0^x f(t)\,dt$

$g(x) = 1 + \int_0^x \left(-\frac{1}{2} + \frac{t^2}{4!} - \frac{t^4}{6!} + \dots\right) dt$

$= 1 + -\frac{1}{2}t + \frac{\frac{1}{3}t^3}{4!} - \frac{\frac{1}{5}t^5}{6!} \Big|_0^x$

$P_5(x) = 1 - \frac{1}{2}x + \frac{x^3}{3\cdot 4!} - \frac{x^5}{5\cdot 6!}$

1pt - 2 terms correct

1pt - all 4 term corrects

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

$P_3(x)$

$\hookrightarrow \left|g(1) - \frac{37}{72}\right| < \frac{1}{6!}$

$P_3(x) = 1 - \frac{1}{2}x + \frac{x^3}{3\cdot 4!}$

$g(1) \approx 1 - \frac{1}{2} + \frac{1}{3\cdot 4!}$

$\approx \frac{37}{72}$

1pt - estimate

$\left|g(1) - \frac{37}{72}\right| <$ max of $|$Remainder of $P_3(x)|$

$< \frac{1}{5\cdot 6!}$

$< \frac{1}{6!}$

1pt- reason

max of Remainder is $\frac{1}{5\cdot 6!}$ b/c abs value of terms will decrease

:)

$1 - \frac{1}{2} + \frac{1}{3\cdot 4\cdot 3\cdot 2}$

$\frac{1}{2} + \frac{1}{72}$

$\frac{36}{72} + \frac{1}{72} = \frac{37}{72}$